Math 451, 01, Exam #1 October 2007
Answer Key

1. (20 points): If the statement is always true, circle "True" and prove it. If the statement is never true, circle "False" and prove that it can never be true. If the statement is true in some cases and false in others, circle "Possible" and give an example and a counter-example.

(a) Let $g \in G$ such that $g^{12} = 1$.
TRUE / POSSIBLE / **FALSE**: $|g| = 5$

If $|g| = 5$, then $g^5 = 1$ so that $g^{12} = g^5 \cdot g^5 \cdot g^2 = 1 \cdot 1 \cdot g^2 = g^2 = 1$. But this means that $|g| \leq 2 < 5 = |g|$ — contradiction!

(b) Define $\varphi : G \to G$ by $\varphi(x) = x^{-1}$.
TRUE / **POSSIBLE** / FALSE: φ is an automorphism of G.

Each element has a unique inverse so that φ is a bijection.
Notice if $\varphi(ab) = \varphi(a)\varphi(b)$, then $b^{-1}a^{-1} = (ab)^{-1} = a^{-1}b^{-1}$. Multiply this equation by ab on the left and ba on the right and get: $ba = ab$. Conversely if $ab = ba$, then $(ab)^{-1} = a^{-1}b^{-1}$ so that $\varphi(ab) = \varphi(a)\varphi(b)$. Therefore, φ is an automorphism if and only if G is abelian.

The statement is true for the trivial group (it's abelian) and the statement is false for S_3 (since it's not abelian).

(c) Suppose that $|G| = 17$.
TRUE / POSSIBLE / FALSE: G is abelian.

The order of G is prime. Therefore, G is cyclic and thus abelian. In fact, we must have that $G \cong \mathbb{Z}_{17}$.

(d) Let H be a normal subgroup of G.
TRUE / **POSSIBLE** / FALSE: $H \times G/H \cong G$

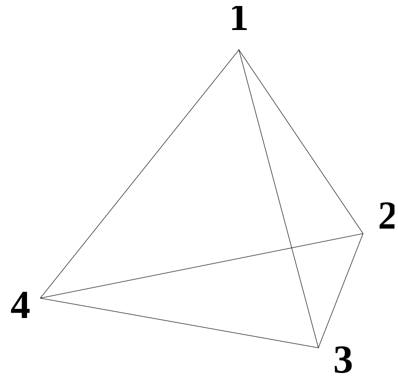
This is true if $G = \mathbb{Z}_6$ and $H = \{0, 2, 4\}$. Then $H \cong \mathbb{Z}_3$ and $G/H \cong \mathbb{Z}_2$. Therefore, $H \times G/H \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6 = G$.

However, this is not true if $G = S_3$ and $H = \{(1), (123), (132)\}$. In this case, $H \cong \mathbb{Z}_3$ and $G/H \cong \mathbb{Z}_2$. But $H \times G/H \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \not\cong S_3 = G$ since the direct product is abelian and the symmetric group is not.

(e) TRUE / POSSIBLE / **FALSE**: S_{10} has an element of order 28.

Remember that if a permutation is written as a product of disjoint cycles, then its order is the least common multiple of the length of those cycles. Thus to achieve order 28 we need either a 28-cycle or a 4-cycle and 7-cycle or a 4-cycle and a 14-cycle etc. The 28-cycle requires at least 28 characters, the 4-cycle and 14-cycle requires at least 18 characters. The best we can do is a 4-cycle and 7-cycle which requires 11 characters. So we see that S_n has no elements of order 28 unless $n \geq 11$.

2. (20 points): Let T be the group of symmetries of the regular tetrahedron.



(a) Label the vertices of T using the numbers 1,2,3,4. We know that T acts on this set. Let $\varphi : T \to S_4$ be the homomorphism corresponding to this group action. Is φ injective? Write out **all** of the elements in the image of φ (*use cycle notation*). Show that $T \cong A_4$.

Let's find the elements in $\varphi(T)$. First, of course, we have the identity: (1). Next, rotating around the line through vertex 1, we get: (234) and (243). Rotating around the line through vertex 2, we get: (134) and (143). Rotating around the line through vertex 3, we get: (124) and (142). Rotating around the line through vertex 4, we get: (123) and (132). The image is a subgroup so it must be closed under compostion. Notice $(123)(143) = (14)(23)$ and $(123)(124) = (13)(24)$ and $(143)(123) = (12)(34)$.

We have found a total of 12 elements so far, but this must be everything since $|T| = 12$ (we can't map onto a set bigger than T). Notice that these elements are exactly the even permuations in S_4.

The image of T is

$$A_4 = \{(1), (234), (243), (134), (143), (124), (142), (123), (132), (14)(23), (13)(24), (12)(34)\}$$

Since the map is one-to-one, we have a faithful action of the tetrahedron on its vertices. Also, since the bijective correspondence between T and A_4 is a homomorphism, they are isomorphic.

(b) Determine the order of each element of A_4, decompose it into its conjugacy classes, and write its class equation.

- $\{(1)\}$ – order 1
- $\{(234), (124), (132), (143)\}$ – all are order 3
- $\{(243), (142), (123), (134)\}$ – all are order 3
- $\{(14)(23), (13)(24), (12)(34)\}$ – all are order 2

The class equation is $12 = 1 + 4 + 4 + 3$.

(c) Find all of the normal subgroups of A_4 and identify each quotient.

$\{(1)\}$ and A_4 are normal subgroups. Now let search for others.

We need to look for unions of conjugacy classes which make up subgroups. Notice that $1 + 4 = 5$, $1 + 4 + 4 = 9$, $1 + 3 + 4 = 8$ don't divide 12. The only possibility is $1 + 3 = 4$.

$H = \{(1), (14)(23), (13)(24), (12)(34)\}$ is a subgroup (since it's closed under composition). It's normal because it's the union of conjugacy classes.

Finally, A_4/A_4 is trivial, $A_4/\{(1)\}$ is just A_4, and A_4/H is isomorphic to $\mathbb{Z}_3$ (since its order is $12/4 = 3$ – a prime – it must be cyclic).

3. (15 points): For each of the following pairs of groups, prove or disprove that they are isomorphic.

(a) $\mathbb{Z}_n \times \mathbb{Z}_2$ and D_n for some $n \geq 3$.

These are not isomorphic. The $\mathbb{Z}_n$ and $\mathbb{Z}_2$ are abelian, so their direct product is too. However, D_n (symmetries of a regular n-gon) is not abelian unless $n = 1$ or $n = 2$.

(b) $\mathbb{R}$ (under addition) and $\mathbb{R}_{>0}$ (under multiplication).

These are isomorphic. Consider the map $\varphi : \mathbb{R} \to \mathbb{R}_{>0}$ defined by $\varphi(x) = e^x$. Notice that $\varphi(x + y) = e^{x+y} = e^x e^y = \varphi(x)\varphi(y)$, so it's a homomorphism. We know that $\ln(x)$ is e^x inverse so it's a bijection (thus an isomorphism).

(c) $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z}$

These are not isomorphic. $\mathbb{Z} \times \mathbb{Z}$ is not cyclic, but $\mathbb{Z}$ is. To see that the direct product isn't cyclic — suppose that (k, ℓ) generates it. But $\langle (k, \ell) \rangle = \{(mk, m\ell)\} \mid m \in \mathbb{Z}\}$. So if this is a generator we must have $(1, 0) = (mk, m\ell)$ for some m, but $mk = 1$ implies either $m = k = -1$ or $m = \ell = 1$. In both cases, we must have $\ell = 0$. But then $(0, 1)$ is not in $\langle (k, \ell) \rangle$. Thus is cannot generate the whole group.

4. (25 points): Let G be a group and let H be a subgroup of G. We define $N = N_G(H) = \{g \in G \mid gHg^{-1} = H\}$ to be the **normalizer** of H in G.

(a) Show that N is a subgroup of G. What does $N = G$ mean?

Notice that $h \in H$ implies that $hHh^{-1} = H$ since H is a subgroup. Therefore, $H \subseteq N$ (that is N is not empty). Next, suppose that $a, b \in N$. This means that $aHa^{-1} = bHb^{-1} = H$. Notice that $(ab)H(ab)^{-1} = a(bHb^{-1})a^{-1} = aHa^{-1} = H$. Thus $ab \in N$. Finally, suppose that $a \in N$ (thus $aHa^{-1} = H$). Then $a^{-1}H(a^{-1})^{-1} = a^{-1}Ha = a^{-1}(aHa^{-1})a$ since $H = aHa^{-1}$. Therefore, $a^{-1}H(a^{-1})^{-1} = H$ so that $a^{-1} \in N$. Therefore, N is a subgroup.

Note: $N = G$ implies that $aHa^{-1} = H$ for **all** $a \in G = N$. This means H is a normal subgroup of G.

(b) Show that H is a normal subgroup of N.

By definition, if $x \in N$, then $xHx^{-1} = H$. Therefore, H is a normal subgroup of N.

(c) Let K be a subgroup of G such that H is a normal subgroup of K. Show that $K \subseteq N$ (this says that N is the biggest subgroup of G in which H is normal).

Let H be a normal subgroup of K. Then $xHx^{-1} = H$ for all $x \in K$. But $xHx^{-1} = H$ implies that $x \in N$. Therefore, $K \subseteq N$.

(d) Let $y \in xN$. Show that the conjugate subgroups xHx^{-1} and yHy^{-1} are equal. How many (distinct) subgroups are conjugate to H?

If $y \in xN$, then $x^{-1}y \in N$. Therefore, $(x^{-1}y)H(x^{-1}y)^{-1} = H$ and thus $x^{-1}yHy^{-1}x = H$. Unravelling this — multiply by x on left and x^{-1} on the right — we get that $yHy^{-1} = xHx^{-1}$.

Notice that the converse is also true, if $yHy^{-1} = xHx^{-1}$, then $(x^{-1}y)H(x^{-1}y)^{-1} = H$ so that $x^{-1}y \in N$. Therefore, $y \in xN$. This means that distinct cosets of N give distinct subgroups conjugate to H. So the number of cosets of N in G gives the number of conjugates of H in G.

Answer: The number of conjugates of H in G is $[G : N_G(H)]$.

5. (20 points): Let M be a *proper* ($\neq G$) subgroup of G. M is a **maximal normal** subgroup if given any other normal subgroup H such that $M \subset H$ but $H \neq M$, then $H = G$. Likewise, M is a **maximal** subgroup if given any other subgroup H such that $M \subset H$ but $H \neq M$, then $H = G$.

(a) Find all of the subgroups of $\mathbb{Z}_6$. Identify which of these are maximal and maximal normal. Also, identify all quotients. Looking at a maximal normal subgroup of $\mathbb{Z}_n$ and then maximal normal subgroup of that subgroup and so on, what is going on?

Since $\mathbb{Z}_6$ is cyclic, it has exactly 1 subgroup of each order dividing 6. These are:
- $\langle 0 \rangle = \{0\}$ – order 1
- $\langle 3 \rangle = \{0, 3\}$ – order 2
- $\langle 2 \rangle = \{0, 2, 4\}$ – order 3
- $\langle 1 \rangle = \mathbb{Z}_6$ – order 6

First, note all subgroups of an abelian group are normal.

Next, notice that the only subgroup containing $\langle 3 \rangle$ is $\mathbb{Z}_6$ itself. The same is true for $\langle 2 \rangle$. Therefore, these are the maximal (& maximal normal) subgroups.

Quotients:
- $\mathbb{Z}_6/\{0\} \cong= \mathbb{Z}_6$ (itself)
- $\mathbb{Z}_6/\langle 3 \rangle \cong \mathbb{Z}_3$
- $\mathbb{Z}_6/\langle 2 \rangle \cong \mathbb{Z}_2$.
- $\mathbb{Z}_6/\mathbb{Z}_6 \cong \{0\}$ (the trivial group)

For $\langle 3 \rangle$ we get the following sequence: $\mathbb{Z}_6 \geq \langle 3 \rangle \geq \{0\}$ whose quotients are isomorphic to $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

For $\langle 2 \rangle$ we get the sequence: $\mathbb{Z}_6 \geq \langle 2 \rangle \geq \{0\}$ whose quotients are isomorphic to $\mathbb{Z}_3$ and $\mathbb{Z}_2$.

This corresponds to the two different ways of factoring 6 into primes — $6 = 2\cdot 3 = 3\cdot 2$.

In general, a maximal normal subgroup of $\mathbb{Z}_n$ will give a quotient isomorphic to $\mathbb{Z}_p$ for some prime p dividing n. Picking a maximal normal subgroup of that subgroup will pick off another prime and so forth. We will end up with a sequence of normal subgroups whose factors are cyclic of prime order. By doing this, we are essentially just factoring n into its prime factors.

Example: $\mathbb{Z}_{60} \geq \langle 2 \rangle \geq \langle 4 \rangle \geq \langle 12 \rangle \geq \langle 0 \rangle$ each subgroup is maximal normal in the previous one. The quotients are: $\mathbb{Z}_2$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_5$ — $2 \cdot 2 \cdot 3 \cdot 5 = 60$.

(b) Let M be a maximal normal subgroup of G. Show that G/M is simple.

Recall that a group is simple if it is non-trivial and it has no non-trivial proper normal subgroups.

Notice that $M \neq G$ so that $G/M \ncong \{1\}$ (thus non-trivial).

Let $\mathcal{H}$ be a normal subgroup of G/M. Then by one of our correspondence theorems we know that $\mathcal{H} = H/M$ for some normal subgroup H of G such that $M \subseteq H$. However, M is maximal normal. Therefore, either $H = M$ (in this case $H/M = M/M = \{M\}$ – the trivial subgroup) or $H = G$ (which means $H/M = G/M$ – the whole quotient). Therefore, G/M has no non-trivial proper normal subgroups, so it is simple.

Bonus: If M is a maximal subgroup of G which is also a normal subgroup, show that in fact G/M is cyclic of prime order (abelian simple).

Notice that G/M is non-trivial since $G \neq M$. Also, G/M has no non-trivial proper subgroups because any subgroup $\mathcal{H}$ of G/M is of the form $\mathcal{H} = H/M$ for some subgroup H of G where $M \subseteq H$, but then either $H = M$ ($\mathcal{H}$ trivial) or $H = G$ ($\mathcal{H} = G/M$).

Let $K \ncong \{1\}$ be any group with no non-trivial proper subgroups. First, notice that K cannot be infinite. If it was, any element of finite order (other than the identity) would generate a non-trivial proper normal subgroup. If there were no elements of finite order, there would be an element of infinite order, call it x. Notice then that $\langle x^2 \rangle$ is then a non-trivial proper normal subgroup (it doesn't contain x). So K must be finite. By Cauchy's theorem if p is a prime dividing the order of K, then there exists some $g \in K$ such that $|g| = p$. But $\langle g \rangle$ is a non-trivial subgroup. Therefore, $\langle g \rangle = K$. So K is cyclic of prime order.

Now apply this discussion to G/M and we are done.