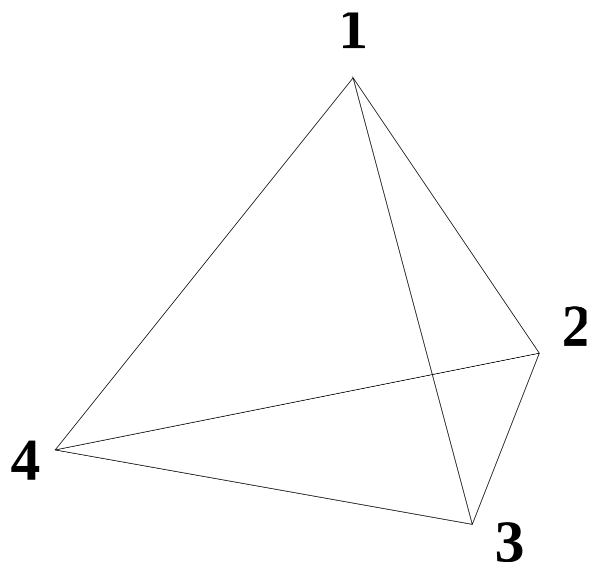
Name ____________________________________ Math 451, 01, Exam #1   Oct., 2007

Be sure to show all your work. Unsupported answers will receive no credit.

Use the backs of the exam pages for scratchwork or for continuation of your answers, if necessary.

Unless otherwise indicated, G is a group with identity 1 whose binary operation is denoted by juxtaposition (i.e. $(a, b) \mapsto ab$).



Problem No.	**Pts Possible**	**Points**
1	20	
2	20	
3	15	
4	25	
5	20	
Bonus	–	
Total	100	

1. (20 points): If the statement is always true, circle "True" and prove it. If the statement is never true, circle "False" and prove that it can never be true. If the statement is true in some cases and false in others, circle "Possible" and give an example and a counter-example.

(a) Let $g \in G$ such that $g^{12} = 1$.
TRUE / POSSIBLE / FALSE: $|g| = 5$

(b) Define $\varphi : G \to G$ by $\varphi(x) = x^{-1}$.
TRUE / POSSIBLE / FALSE: φ is an automorphism of G.

(c) Suppose that $|G| = 17$.
TRUE / POSSIBLE / FALSE: G is abelian.

(d) Let H be a normal subgroup of G.
TRUE / POSSIBLE / FALSE: $H \times G/H \cong G$

(e) **TRUE / POSSIBLE / FALSE**: S_{10} has an element of order 28.

2. (20 points): Let T be the group of symmetries of the regular tetrahedron.

(a) Label the vertices of T using the numbers 1,2,3,4. We know that T acts on this set. Let $\varphi : T \to S_4$ be the homomorphism corresponding to this group action. Is φ injective? Write out **all** of the elements in the image of φ (*use cycle notation*). Show that $T \cong A_4$.

(b) Determine the order of each element of A_4, decompose it into its conjugacy classes, and write its class equation.

(c) Find all of the normal subgroups of A_4 and identify each quotient.

3. (15 points): For each of the following pairs of groups, prove or disprove that they are isomorphic.

(a) $\mathbb{Z}_n \times \mathbb{Z}_2$ and D_n for some $n \geq 3$.

(b) $\mathbb{R}$ (under addition) and $\mathbb{R}_{>0}$ (under multiplication).

(c) $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z}$

4. (25 points): Let G be a group and let H be a subgroup of G. We define $N = N_G(H) = \{g \in G \mid gHg^{-1} = H\}$ to be the **normalizer** of H in G.

(a) Show that N is a subgroup of G. What does $N = G$ mean?

(b) Show that H is a normal subgroup of N.

(c) Let K be a subgroup of G such that H is a normal subgroup of K. Show that $K \subseteq N$ (this says that N is the biggest subgroup of G in which H is normal).

(d) Let $y \in xN$. Show that the conjugate subgroups xHx^{-1} and yHy^{-1} are equal. How many (distinct) subgroups are conjugate to H?

5. (20 points): Let M be a *proper* ($\neq G$) subgroup of G. M is a **maximal normal** subgroup if given any other normal subgroup H such that $M \subset H$ but $H \neq M$, then $H = G$. Likewise, M is a **maximal** subgroup if given any other subgroup H such that $M \subset H$ but $H \neq M$, then $H = G$.

(a) Find all of the subgroups of $\mathbb{Z}_6$. Identify which of these are maximal and maximal normal. Also, identify all quotients. Looking at a maximal normal subgroup of $\mathbb{Z}_n$ and then maximal normal subgroup of that subgroup and so on, what is going on?

(b) Let M be a maximal normal subgroup of G. Show that G/M is simple.

Bonus: If M is a maximal subgroup of G which is also a normal subgroup, show that in fact G/M is cyclic of prime order (abelian simple).