Name ______________________________ Math 451, 01, Exam #2 Nov., 2007

Be sure to show all your work. Unsupported answers will receive no credit.

Use the backs of the exam pages for scratchwork or for continuation of your answers, if necessary.

Unless otherwise indicated, G is a group with identity 1 whose binary operation is denoted by juxtaposition (i.e. $(a, b) \mapsto ab$).

Problem No.	**Pts Possible**	**Points**
1	25	
2	20	
3	20	
4	25	
5	15	
Total	100	

1. (25 points): If the statement is always true, circle "True" and prove it. If the statement is never true, circle "False" and prove that it can never be true. If the statement is true in some cases and false in others, circle "Possible" and give an example and a counter-example.

(a) Let G be an abelian group of order 16.
TRUE / **POSSIBLE** / **FALSE**: The class equation of G is:
$$16 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 2 + 2 + 4$$

(b) Let G be a non-abelian group of order 14.
TRUE / **POSSIBLE** / **FALSE**: The class equation of G is:
$$14 = 1 + 2 + 2 + 2 + 7$$

(c) Let G be a non-abelian simple group.
TRUE / **POSSIBLE** / **FALSE**: G has a subgroup of index 4.

(d) Let $G = \langle x, y \mid R \rangle$ where R is a set of 3 relations.
TRUE / **POSSIBLE** / **FALSE**: G is finite.

(e) **TRUE** / **POSSIBLE** / **FALSE**: $\mathrm{PSL}_2(\mathbb{F}_9)$ has at least 24 elements of order 5.

2. (20 points): Let p be a prime and k a positive integer.

(a) Show that a group of order p^k has a non-trivial center. Then explain why a group of order p^k is simple if and only if $k = 1$.

(b) Show that groups of order p^2 are abelian. Then classify the groups of order p^2.

3. (20 points): Classify.

(a) Classify the groups of order $2p$ where p is an odd prime.

(b) Classify the groups of order 99.

4. (25 points): William Wallace would be proud.

(a) State the universal property of a free group F generated by the set X equipped with mapping $i : X \to F$.

(b) Sketch the proof of: If F_1 and F_2 are free on X, then $F_1 \cong F_2$.

(c) Identify the free groups on 0 and 1 generator, then use the universal property to prove that F is not abelian when $|X| > 1$.

5. (15 points): Choices: Choose **one** of the following problems.

I. Use the Todd-Coxeter algorithm to find a permutation representation of the group with presentation:

$$\langle x, y \mid x^2, y^2, xyx^{-1}y^{-1} \rangle$$

What is the name of this group?

II. Derive the formula for the order of the group $\mathrm{PSL}_2(\mathbb{F}_q)$ where $\mathbb{F}_q$ is the finite field of order q.