Math 4720 & 5210 # Test #1: Group Theory **Due: October 5^{th}, 2010**

Blanket Assumption: G are G' are groups with identity 1.
Do not assume groups are finite unless told otherwise!!!

Bill's Big Mistake: Some parts of problems have answers which are worked out in my old Rutgers exam answer keys – oh well – free points for "exam review".

1. (35 points): Oh no! Is it always true? Possibly true? Or always false?

For each of the following statements. If the statement is always true, answer "TRUE" and prove that this is the case. If the statement is true for some cases and false for others, then answer "POSSIBLE" and give an example (of the statement holding) and a counter-example (to show it can fail). If the statement can never be true, answer "FALSE" and prove that it is impossible for the statement to hold.

(a) Let $g \in G$ such that $g^{15} = 1$ and $g^6 = 1$. Is it true, possible, or false that $|g| = 5$?

(b) True or false, S_7 has an element of order 10.

(c) Let $|G| = 10$ where G is a non-abelian group. Is it true, possible, or false that the class equation for G is $10 = 1 + 2 + 2 + 5$?

(d) Let $|G| = 16$. Is it true, possible, or false that the class equation for G is $16 = 1 + 1 + 1 + 1 + 1 + 1 + 2 + 2 + 2 + 4$?

(e) Suppose $|G| = 12$, S be a set of 8 elements, and G acts on S. Is it true, possible, or false that G acts transitively on S?

(f) Let $D = \{(x, x) \,|\, x \in G\}$ (the diagonal sub**set** of $G \times G$). Is it true, possible, or false that $(G \times G)/D \cong G$?

(g) $G = \langle x, y \,|\, R\rangle$ and R is a set of 4 relations. Is it true, possible, or false that G is finite?

(h) [For 5210 students] Let G be a non-abelian simple group with a subgroup of index 5. Is it true, possible, or false that $G \cong A_5$?
Fact/Hint: A_5 is the only non-abelian group (up to isomorphism) with order < 100.

2. (10 points): Maximal subgroups and the subgroup lattice.

Recall that a proper subgroup of G is any subgroup not equal to G itself.

Definition: We say that a proper subgroup M is a **maximal subgroup** of G if given any subgroup H such that $M \subseteq H \subseteq G$, then either $H = M$ or $H = G$. In other words, there are no subgroups "in-between" M and G.

Definition: We say that a proper normal subgroup M is a **maximal normal subgroup** of G if given any $H \triangleleft G$ such that $M \subseteq H \subseteq G$, then either $H = M$ or $H = G$. So there are no *normal* subgroups "in-between" M and G.

(a) Explain why $\mathbb{Z}_3 \times \mathbb{Z}_3$ is **not** cyclic. Then explain why all of its proper subgroups are cyclic. Finally, draw a subgroup lattice for $\mathbb{Z}_3 \times \mathbb{Z}_3$.

(b) Let M be a maximal normal subgroup of G. Prove that G/M is a simple group.

(c) [For 5210 students] What are the maximal subgroups of $\mathbb{Z}_n$? I suggest working out a few concrete examples before **proving** your conjecture. Relate your answer to part (b) and the fact that the only abelian simple groups are $\mathbb{Z}_p$ (p prime).

3. (10 points): "You're free cheesy bread!"

(a) $G = \langle x \,|\, x = x^{-2}\rangle$. List all of the elements of G. What is G more commonly known as?

(b) Let $F(\{a,b\})$ be the free group on 2 generators. Show $|aba^{-1}b^{-1}| = \infty$ in $F(\{a,b\})$.

Note: You must use the universal property. Hint: Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ in $\mathrm{GL}_2(\mathbb{Q})$.

4. (5 points): A classy problem.

(a) Briefly explain why normal subgroups are unions of conjugacy classes. Partition $D_{10} = \langle r, s \mid r^5 = 1, s^2 = 1, rsrs = 1 \rangle$ into conjugacy classes. Find all of the normal subgroups of D_{10}.

(b) [For 5210 students] Find the conjugacy classes and the class equation for D_{2p} where p is an odd prime. Then use this to find all of the normal subgroups of D_{2p}.

5. (10 points): Random stuff.

(a) Let G be a group of order 60. Show $|Z(G)| \neq 4$. **Hint:** 60 divided by 4 is 15.

(b) Let S be a subset of G. Recall that $\langle S \rangle$ is the subgroup of G generated by S. Suppose $\varphi : G \to G'$ is a homomorphism. Prove that $\varphi(\langle S \rangle) = \langle \varphi(S) \rangle$.

6. (15 points): There are no **non-abelian** simple groups of order less than 60. Prove this. If you use a theorem to eliminate a collection of orders, please state the theorem and list of the corresponding orders eliminated. For example:

- **Definition:** Simple groups are non-trivial groups with no proper non-trivial subgroups.
  This eliminates $|G| = 1$.
- **Theorem:** Groups of prime order are cyclic. And thus abelian.
  This eliminates $|G| = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59$

NOTE: As mentioned in class, you may only use theorems from class and from the part of the textbook which we've covered and *no Burnside's theorem* (yes it's stated in chapter 4 but it isn't proved until the end of the textbook).

~~1~~	~~2~~	~~3~~	$4 = 2^2$	~~5~~	$6 = 2 \cdot 3$	
~~7~~	$8 = 2^3$	$9 = 3^2$	$10 = 2 \cdot 5$	~~11~~	$12 = 2^2 \cdot 3$	
~~13~~	$14 = 2 \cdot 7$	$15 = 3 \cdot 5$	$16 = 2^4$	~~17~~	$18 = 2 \cdot 3^2$	
~~19~~	$20 = 2^2 \cdot 5$	$21 = 3 \cdot 7$	$22 = 2 \cdot 11$	~~23~~	$24 = 2^3 \cdot 3$	
$25 = 5^2$	$26 = 2 \cdot 13$	$27 = 3^3$	$28 = 2^2 \cdot 7$	~~29~~	$30 = 2 \cdot 3 \cdot 5$	
~~31~~	$32 = 2^5$	$33 = 3 \cdot 11$	$34 = 2 \cdot 17$	$35 = 5 \cdot 7$	$36 = 2^2 \cdot 3^2$	
~~37~~	$38 = 2 \cdot 19$	$39 = 3 \cdot 13$	$40 = 2^3 \cdot 5$	~~41~~	$42 = 2 \cdot 3 \cdot 7$	
~~43~~	$44 = 2^2 \cdot 11$	$45 = 3^2 \cdot 5$	$46 = 2 \cdot 23$	~~47~~	$48 = 2^4 \cdot 3$	
$49 = 7^2$	$50 = 2 \cdot 5^2$	$51 = 3 \cdot 17$	$52 = 2^2 \cdot 13$	~~53~~	$54 = 2 \cdot 3^3$	
$55 = 5 \cdot 11$	$56 = 2^3 \cdot 7$	$57 = 3 \cdot 19$	$58 = 2 \cdot 29$	~~59~~	$\boxed{60 = 2^2 \cdot 3 \cdot 5}$	$\leftarrow A_5$

7. (10 points): Sylow-riffic

(a) Let G be a **simple** group of order $168 = 2^3 \cdot 3 \cdot 7$. Prove G has 48 elements of order 7.

(b) Let G be a group of order 85. Show $G \cong \mathbb{Z}_{85}$.

8. (5 points): Action! [Section 4.2 #7] Let Q be the group of quaternions.

(a) Prove Q is isomorphic to a subgroup of S_8. Write down the elements of this subgroup.

(b) [For 5210 students] Prove that Q is **not** isomorphic to a subgroup of S_k for $k \leq 7$.