Math 4720 & 5210 **Test #1: Group Theory** **Due: Wed., Feb. 27^{th}, 2019**

The Rules: You are to work on your exam alone. Do not discuss these problems with classmates, faculty, or any other person (except Dr. Cook). However, you may consult the voices in your head or ask your dog (if you have a talking dog). You may consult books, notes, and *pre-existing* internet resources. You may not post questions about the test online. However, if a question is already answered somewhere on the vast interwebs, that's fair game. *If you get a solution from somewhere online or in a book, cite your source!*

Blanket Assumption: G, H, K, N are groups or subgroups with identity 1.
Blanket Unassumption: Do not assume groups are finite unless told otherwise!

#1 (15 points) Enter the Matrix Recall that $\mathrm{GL}_m(\mathbb{R})$ is the group of invertible $m \times m$ matrices over the reals (the operation is matrix multiplication). Also, recall that $\mathrm{SL}_m(\mathbb{R}) = \{A \in \mathrm{GL}_m(\mathbb{R} \mid \det(A) = 1\}$.

(a) Show $\mathrm{SL}_m(\mathbb{R})$ is a **normal** subgroup of $\mathrm{GL}_m(\mathbb{R})$. Then identify the quotient $\mathrm{GL}_m(\mathbb{R}) / \mathrm{SL}_m(\mathbb{R})$.
Hint: Determinant map.

(b) We know that $\mathrm{GL}_2(\mathbb{Z}_n)$ is not Abelian for any integer $n > 1$.
Consider $A = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$. Notice that $AB = \begin{bmatrix} 11 & 21 \\ 2 & 4 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 6 \\ 2 & 14 \end{bmatrix}$.
While the above matrices A and B do prove that $\mathrm{GL}_2(\mathbb{Z}_3)$ is not Abelian, these matrices do **not** prove that $\mathrm{GL}(\mathbb{Z}_5)$ is non-Abelian. Why? They also don't help for $\mathrm{GL}_2(\mathbb{Z}_6)$. Why?
Hint: The same "Why?" doesn't work for both $n = 5$ and $n = 6$.

#2 (40 points) True, Possible, False If the statement is always true, answer "TRUE" and prove that this is the case. If the statement is true for some cases and false for others, then answer "POSSIBLE" and give an example (of the statement holding) and a counter-example (to show it can fail). If the statement can never be true, answer "FALSE" and prove that it is impossible for the statement to hold.

(a) Let p be a prime and G be a group of order p^2.
TRUE / POSSIBLE / FALSE: G is cyclic

(b) Suppose G is a group of order 24 whose class equation is $24 = 1+1+1+1+2+2+2+2+3+3+3+3$.
TRUE / POSSIBLE / FALSE: G has a normal subgroup of order 4.

(c) Let D_{12} act on the set $X = \{1, 2, 3, 4, 5\}$.
TRUE / POSSIBLE / FALSE: D_{12} acts transitively.

(d) Let $G = \langle a, b \mid a^3 = 1, b^2 = 1, aba^{-1}b^{-1} = 1, R\rangle$ where R is a set of 0 or more additional relations.
TRUE / POSSIBLE / FALSE: G is finite.

(e) Let G be a group of order 60 with *less* than 24 elements of order 5.
TRUE / POSSIBLE / FALSE: G is not simple.

(f) Let $\tau \in S_{11}$ and let $\sigma = (12)\tau$
TRUE / POSSIBLE / FALSE: The order of σ is 20.

(g) Suppose $g \in G$ and $g^{12} = 1$ and $g^7 = 1$.
TRUE / POSSIBLE / FALSE: $g = 1$.

(h) Let G be a non-abelian group of order $260 = 2^2 \cdot 5 \cdot 13$.
TRUE / POSSIBLE / FALSE: G has a cyclic subgroup of order 13.

(i) [For 5210 students] Let G be a non-abelian simple group with a subgroup of index 5.
TRUE / POSSIBLE / FALSE: $G \cong A_5$
Fact/Hint: A_5 is the only non-abelian group (up to isomorphism) with order < 100.

#3 (15 points) Dodecaproblem Computing in $D_{12} = \langle x, y \mid x^{12} = 1, y^2 = 1, xyxy = 1\rangle$.

(a) Simplify $x^3y^3x^{22}yx^{-2}y^6$ by writing it in the form x^jy^k for some non-negative integers j, k (where j, k are as small as possible).

(b) Find the conjugacy classes of D_{12} and write down its class equation.

(c) Does the class equation allow for a normal subgroup of order 6? Why or why not?

#4 (20 points) Basic Randos A few proofs.

(a) Let $N \triangleleft G$ and H be any subgroup. Show $N \cap H \triangleleft H$.

(b) Let $x \in G$. Show x and x^{-1} have the same order. (Do not assume the order of x is finite.)

(c) Let $S \subseteq G$ (subset not subgroup) and suppose $\varphi : G \to K$ is a homomorphism. Show $\varphi(\langle S \rangle) = \langle \varphi(S) \rangle$ (the image of the subgroup generated by S is generated by the image of S).

#5 (20 points) Conjugate and Factor Away Your Troubles A symmetric group and quotient problem.

Note: $S_4 = \{(1), (12), (13), (14), (23), (24), (34), (123), (132), (124), (142), (134), (143), (234), (243),$
$(1234), (1243), (1324), (1342), (1423), (1432), (12)(34), (13)(24), (14)(23)\}$

(a) $H = \{(1), (12)(34), (13)(24), (14)(23)\}$ is a subgroup of S_4. Using what you know about the conjugacy classes of S_4, in a few words explain why H is normal.

(b) List the elements of and write down a Cayley table for S_4/H.

(c) Find the order of each element of S_4/H. (I am asking for the order as a group element. I am not asking for the cardinality of these sets.)

(d) Is S_4/H Abelian? Cyclic? Explain your answers.

#6 (20 points) Like a Piano – Ok, Not Really Our friends D_{88} and $\mathbb{Z}_{88}$. Note that $88 = 2^3 \cdot 11$.

(a) Make a table listing the orders of the elements of $\mathbb{Z}_{88}$ along with the number of such elements. Then make a corresponding table for D_{88}.

(b) Draw the subgroup lattice for $\mathbb{Z}_{88}$.

(c) Draw the subgroup lattice for $\mathbb{Z}_{88}/\langle 22 \rangle$.

#7 (15 points) It's a Sylow Party! What do the Sylow theorems say about G (including the number of Sylow subgroups, number of elements of various orders, could a group of that order be simple) if....

(a) $|G| = 7 \cdot 19 = 133$?

(b) $|G| = 3^2 \cdot 7 = 63$?

(c) $|G| = 2^3 \cdot 3 \cdot 7 = 168$?

#8 (15 points) Maximizing Your Difficulties Maximal subgroups and the subgroup lattice.

Recall that a proper subgroup of G is any subgroup not equal to G itself.

Definition: We say that a proper subgroup M is a **maximal subgroup** of G if given any subgroup H such that $M \subseteq H \subseteq G$, then either $H = M$ or $H = G$. In other words, there are no subgroups "between" M and G.

(a) Explain why $\mathbb{Z}_3 \times \mathbb{Z}_3$ is **not** cyclic. Then explain why all of its proper subgroups are cyclic. Finally, draw a subgroup lattice for $\mathbb{Z}_3 \times \mathbb{Z}_3$.

(b) Explain why a maximal subgroup of non-abelian simple group must be self-normalizing. [This means that if G is non-abelian simple and M is maximal, then $N_G(M) = M$.]

(c) [For 5210 students] What are the maximal subgroups of $\mathbb{Z}_n$? I suggest working out a few concrete examples before **proving** your conjecture.

#9 (15 points) Quaternions Adventures Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the group of quaternions.

(a) Recall that all of the proper subgroups of Q are cyclic. Let X be the set of (cyclic) subgroups of order 4 and their (left) cosets. Also, let Q act on X by left multiplication: $g \cdot xH = gxH$. Briefly explain why this is an action. Then find the orbits of each element and the corresponding stabilizers.

(b) [For 5210 students] Prove that Q is **not** isomorphic to a subgroup of S_k for $k \leq 7$. Follow up, what tells us that Q is isomorphic to a subgroup of S_8? In other words, the smallest symmetric group which allows for a subgroup isomorphic to Q is S_8.

Hint: What does being isomorphic to a subgroup of S_k mean in terms of an action? What is the intersection of all stabilizers? Consider possible orbit and stabilizer sizes.

Is this also true for D_4?

#10 (25 points) There are no **non-abelian** simple groups of order less than 60. Prove this.

If you use a theorem to eliminate a collection of orders, please state the theorem and list of the corresponding orders eliminated. You may only use theorems from class!!

For example, my solution might begin with...

- By definition simple groups are non-trivial groups. This eliminates $|G| = 1$.
- We have a theorem which states that groups of prime order are cyclic and thus also abelian. This eliminates $|G| = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59$

Note: As mentioned in class, you may only use theorems from class and from the part of the textbook which we've covered. So, for example, you may **not** use *Burnside's theorem* (this theorem rules out all groups of orders $p^k q^\ell$).

~~1~~	~~2~~	~~3~~	$4 = 2^2$	~~5~~	$6 = 2 \cdot 3$	
~~7~~	$8 = 2^3$	$9 = 3^2$	$10 = 2 \cdot 5$	~~11~~	$12 = 2^2 \cdot 3$	
~~13~~	$14 = 2 \cdot 7$	$15 = 3 \cdot 5$	$16 = 2^4$	~~17~~	$18 = 2 \cdot 3^2$	
~~19~~	$20 = 2^2 \cdot 5$	$21 = 3 \cdot 7$	$22 = 2 \cdot 11$	~~23~~	$24 = 2^3 \cdot 3$	
$25 = 5^2$	$26 = 2 \cdot 13$	$27 = 3^3$	$28 = 2^2 \cdot 7$	~~29~~	$30 = 2 \cdot 3 \cdot 5$	
~~31~~	$32 = 2^5$	$33 = 3 \cdot 11$	$34 = 2 \cdot 17$	$35 = 5 \cdot 7$	$36 = 2^2 \cdot 3^2$	
~~37~~	$38 = 2 \cdot 19$	$39 = 3 \cdot 13$	$40 = 2^3 \cdot 5$	~~41~~	$42 = 2 \cdot 3 \cdot 7$	
~~43~~	$44 = 2^2 \cdot 11$	$45 = 3^2 \cdot 5$	$46 = 2 \cdot 23$	~~47~~	$48 = 2^4 \cdot 3$	
$49 = 7^2$	$50 = 2 \cdot 5^2$	$51 = 3 \cdot 17$	$52 = 2^2 \cdot 13$	~~53~~	$54 = 2 \cdot 3^3$	
$55 = 5 \cdot 11$	$56 = 2^3 \cdot 7$	$57 = 3 \cdot 19$	$58 = 2 \cdot 29$	~~59~~	$\boxed{60 = 2^2 \cdot 3 \cdot 5}$	$\leftarrow A_5$