Math 4720 & 5210 **Test #2: Ring Theory** **Due: April 17^{th}, 2019**

Name: ____________________________________ SHOW YOUR WORK!!

1. ($12 \times 5 = 60$ points): If the statement is always true, write "True" and prove it. If the statement is never true, write "False" and prove that it can never be true. If the statement is true in some cases and false in others, write "Possible" then give an example and a counter-example.

(a) Let $n \in \mathbb{Z}$ and $n > 1$.
TRUE / POSSIBLE / FALSE: Both cancellation laws hold in $\mathbb{Z}_n$.

(b) Let $f(x) \in \mathbb{Q}[x]$ be a reducible polynomial of degree 5 with no roots in $\mathbb{Q}$.
TRUE / POSSIBLE / FALSE: $f(x)$ has no irreducible factor of degree 3.

(c) Let R and S be commutative rings with 1. In addition, $\text{char}(R) = 3$ and $\text{char}(S) = 2$.
TRUE / POSSIBLE / FALSE: $R \times S$ has characteristic 6.

(d) Let R be a sub**ring** of $\mathbb{C}$.
TRUE / POSSIBLE / FALSE: $f(x) = 6x + 2$ is irreducible in $R[x]$.

(e) Let R be a commutative ring with 1 and $I \triangleleft R$ a prime ideal such that R/I is finite.
TRUE / POSSIBLE / FALSE: I is maximal.

(f) $n \in \mathbb{Z}$ and $n > 1$.
TRUE / POSSIBLE / FALSE: $x^3 + 1$ is irreducible in $\mathbb{Z}_n[x]$.

(g) $n \in \mathbb{Z}$ and $n > 1$.
TRUE / POSSIBLE / FALSE: $\mathbb{Z}_n[x]$ is a UFD but not a Euclidean domain.

(h) Let $\mathbb{F}$ be a subfield of $\mathbb{C}$, $f(x) = x^2 - 5 \in \mathbb{F}[x]$, and $I = (f(x)) \triangleleft \mathbb{F}[x]$.
TRUE / POSSIBLE / FALSE: $\mathbb{F}[x]/I$ is a field.

(i) Let $f(x) = a(x)b(x)c(x)$ be a factorization in $\mathbb{Q}[x]$ of $f(x)$ into irreducibles such that $a(x), b(x), c(x)$ are distinct factors (they are not associates). Moreover, let $I = (f(x))$.
TRUE / POSSIBLE / FALSE: $\mathbb{Q}[x]/I$ is a direct product of fields.

(j) Let $\mathbb{F}$ be a subfield of $\mathbb{R}$, and $f(x) \in \mathbb{F}[x]$.
TRUE / POSSIBLE / FALSE: $f(x)$ is irreducible in $\mathbb{R}[x]$ but not in $\mathbb{F}[x]$.

(k) Let R be a sub**ring** (with 1) of $\mathbb{C}$.
TRUE / POSSIBLE / FALSE: 4 is irreducible.

(l) Let R be a ring with 1 whose characteristic is 7.
TRUE / POSSIBLE / FALSE: R is finite.

2. (35 points): Isomorphic or not?
Decide whether the following pairs of rings are isomorphic or not. Justify your answer.

(a) $(\mathbb{Z}[x,y])^{2\times 2}$ (2×2 matrices with entries in $\mathbb{Z}[x,y]$) and $\mathbb{Q}(x)$ (rational functions)

(b) The fraction field of $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Q}[x]/(x^2-2)$

(c) $\mathbb{Z}[i]$ and $\mathbb{Q}[x,y,z]$

(d) $\mathbb{Z}_3$ and $\mathbb{Z}[x]/(x,3)$

(e) $\mathbb{Z}_6[x]$ and $\mathbb{Z}_4[x,y]$

(f) For this last part, consider the ring of Gaussian integers $\mathbb{Z}[i]$ and the ring of integers $\mathbb{Z}$. Obviously these rings are not isomorphic. However, they have many matching (ring) properties [For example, they're both commutative]. List as many matching properties as you can. Also, give *at least* one property which doesn't match (showing that they cannot be isomorphic).

3. (10 points): Show $\mathbb{Q}[\sqrt{3}]$ is a sub**field** of $\mathbb{R}$. (First show it is a subring, then show it is a field.)

4. (15 points): Give examples of **non-zero proper** ideals in $\mathbb{Z}[x]$ which are (i) maximal, (ii) prime but not maximal, and (iii) neither prime nor maximal. Can we find similar examples if we switch to $\mathbb{Q}[x]$?

5. (30 points): Isomorphic fun! ◡̈

(a) Let $S = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$. Show that S is a subring of $\mathbb{R}^{2\times 2}$ (2×2 real matrices) which is isomorphic to $\mathbb{C}$ (the field of complex numbers).

(b) Prove $\mathbb{Q}[x] / (x^2 - 1) \cong \mathbb{Q} \times \mathbb{Q}$ without using Chinese remaindering. Specifically use the map defined by $\varphi(f(x)) = (f(-1), f(1))$ and apply the first isomorphism theorem.

6. (10 points): Let $\mathbb{F}$ be a field. First, explain why the only ideals of $\mathbb{F}$ are $\{0\}$ and $\mathbb{F}$ itself. Then show that if $\varphi : \mathbb{F} \to R$ is a homomorphism, then either $\varphi(x) = 0$ for all $x \in \mathbb{F}$ or φ is injective (i.e. one-to-one).

7. (35 points): Workin' mod n.

(a) Factor $x^8 - 1$ in $\mathbb{Z}_2[x]$.

(b) Give **examples** of irreducible polynomials of degrees 1, 2, and 3 in $\mathbb{Z}_5[x]$ (don't try to list out *all* of the irreducibles).

(c) Construct a field of order 9, call it $\mathbb{F}_9$. Write down the addition and multiplication tables for $\mathbb{F}_9$.

8. (15 points): Let R be a commutative ring with 1. Recall that R is Artinian if it has the descending chain condition on ideals (DCC). This means that given ideals $I_k \triangleleft R$ such that $I_1 \supseteq I_2 \supseteq \cdots$ there exists some positive integer N such that $I_N = I_{N+1} = I_{N+2} = \cdots$ (every descending chain of ideals eventually stabilizes).

Show that if R is Artinian, then R is classical (every element of R is either zero, a zero divisor, or a unit). *Hint:* Let $x \in R$. Suppose that x is not zero and not a unit. Consider $R \supseteq (x) \supseteq (x^2) \supseteq \cdots$.

Pointless note: Instead of Artinian (i.e., our ring has the DCC) we could just assume descending chain condition on principal ideals (DCCP) holds.

Problems for 5210 Students.

9. (15 points): Let R be a principal ideal domain and $\{0\} \neq I \triangleleft R$. Show that R/I is both Noetherian and Artinian (i.e., R/I has both the ascending and descending chain conditions on ideals).

10. (20 points): Let R be a commutative ring with $1 \neq 0$. We say that R is a *local ring* if R has a unique maximal ideal (i.e. R has exactly one maximal ideal).

(a) Show that if R is local with maximal ideal M, then $R - M = \{x \in R \mid x \notin M\} = R^\times$ – that is – the units of R are exactly the elements which are outside of M.

(b) Now let $M = R - R^\times$ (the collection of non-units). Show that if M is an ideal, then R is a local ring with maximal ideal M – that is – if the non-units form an ideal, then that ideal is the unique maximal ideal of R so that R is a local ring.

11. (15 points): You constructed a field of order 9 (i.e., $\mathbb{F}_9$) in 7(c). Consider $Y^3 + Y + 2 \in \mathbb{F}_9[Y]$.

(a) Is $Y^3 + Y + 2$ irreducible in $\mathbb{F}_9[Y]$? Why or why not?

(b) Identify the ring $\mathbb{F}_9[Y] / (Y^3 + Y + 2)$