Math 5230

Almost Final Exam

Due by Monday at 5pm (Dec. 5^{th})

Extended Edition

DIRECTIONS: You may use books, notes, software, and existing online resources to complete this exam. You may **not** talk to anyone (except me) about these problems or ask for help online.

NOTATION: Recall that $P_n = \{a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0 \mid a_0, \ldots, a_n \in \mathbb{R}\}$ is the space of all polynomials of degree at most n, $\text{Hom}(V, W) = \mathcal{L}(V, W)$ is the space of all linear transformations from V to W, and $\mathbb{R}[x]$ is the space of all polynomials with real coefficients.

1. Let $A = \begin{bmatrix} 2 & 1 & 2 & 4 & 6 \\ 0 & 0 & 2 & 0 & 2 \\ 2 & 1 & 4 & 6 & 8 \end{bmatrix}$.

(a) Find an invertible matrix P such that $PA = R$ is the reduced row echelon form of A.

(b) Write P as a product of elementary matrices.

2. Let $S_1 = \{x^3 - 1, x^2 + 2x, 2x^3 - x^2 - 2x - 2\}$. Define $W = \{ax^3 + bx^2 + cx + d \mid a + 2b - c + d = 0\}$.

(a) Show W is a subspace.

(b) Find a basis for W.

(c) Show $\text{span}(S_1) \subseteq W$. Are they equal? Why or why not?

3. Let $T : P_2 \to \mathbb{R}^{2\times 3}$ be defined by $T(ax^2 + bx + c) = \begin{bmatrix} a & b & c \\ b & c & a \end{bmatrix}$. Let $\beta_1 = \{1, x, x^2\}$ (note the order) and $\beta_2 = \{1 - x, 1 + x, x - x^2\}$. Also, let $\gamma_1 = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$ and

$$\gamma_2 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$

(a) Prove T is linear.

(b) Prove that β_2 is a basis for P_2 and γ_2 is a basis for $\mathbb{R}^{2\times 3}$.

(c) Find $[T]_{\beta_1}^{\gamma_1}$.

(d) Find a basis for the kernel and range of T. State T's rank and nullity. Is T one-to-one, onto, both or neither?

(e) Find the change of coordinate matrix from β_1 to β_2.

(f) Find $[T]_{\beta_2}^{\gamma_2}$.

(g) Find $[x^2 + 2x + 1]_{\beta_2}$. Then compute $[T(x^2 + 2x + 1)]_{\gamma_2}$ using part ~~(e)~~ (f) [Actually, (e) is useful too.].

4. Let $A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$.

(a) Looking at A, we can immediately conclude it is diagonalizable over the real numbers. Why?

(b) Find the characteristic polynomial of A and A's eigenvalues as well as their algebraic and geometric multiplicities. [Note: You probably will want to have technology factor your polynomial for you.]

(c) Find an **orthogonal** matrix P that diagonalizes A.

5. We run an experiment and find the following data points: $(x, y, z) = (1, 2, 3)$, $(0, -1, 2)$, $(2, 1, 1)$, and $(-1, 1, 0)$. Let $f(x, y) = A\sin(xy) + B\cos(x + y) + C$. Find constants A, B, C such that $z = f(x, y)$ best fits the given data (give a least squares solution). [Note: Doing this by hand would be a very very bad idea. You can give an approximate solution (i.e. use decimal approximations of A, B, C).]

6. Let $T, U : V \to W$ be a linear transformations between vector spaces V and W (vector spaces over a field $\mathbb{F}$).

(a) Let $S \subseteq W$ and suppose S is linearly independent. Show that $T^{-1}(S) = \{\mathbf{v} \in V \mid T(\mathbf{v}) \in S\}$ is linearly independent.

(b) Recall that the annihilator of a subset $X \subseteq V$ is $A(X) = \{f \in V^* \mid f(\mathbf{x}) = 0 \text{ for all } \mathbf{x} \in X\}$ and $T^t : W^* \to V^*$ is defined by $T^t(f) = f \circ T$. Show that $A(T(V)) = \mathrm{Ker}(T^t)$ (i.e. the annihilator of the range is equal to the kernel of the transpose map).

(c) Suppose that neither T nor U are the zero transformation. Also, assume that $T(V) \cap U(V) = \{\mathbf{0}\}$ (i.e. the ranges of T and U) have a trivial intersection. Prove that $\{T, U\}$ is linearly independent as a subset of $\mathcal{L}(V, W)$.

(d) Suppose that V and W are finite dimensional and use a previous part to show that T is an isomorphism if and only if T^t is an isomorphism.

(e) Let $U, T : V \to V$ be commuting linear operators (i.e. $U \circ T = T \circ U$). Show that any eigenspace of U is T-invariant.

7. Let T be a linear operator defined on a finite dimensional inner product space. Show $T^*(V) \subseteq \mathrm{Ker}(T)^\perp$ (the range of the adjoint is contained in the orthogonal complement of the kernel of the operator). Bonus: Show equality.

8. If the statement is always true, write "True" and **prove it**. If the statement is never true, write "False" and **prove** that it can never be true. If the statement is true in some cases and false in others, write "Possible" then give an *example* **and** a *counter-example*.

You may skip 4.

(a) Let $A \in \mathbb{R}^{5\times 5}$ be a matrix with characteristic polynomial $f(t) = t(t-1)(t-2)(t-3)(t-4)$.
TRUE / POSSIBLE / FALSE: A is diagonalizable.

(b) Let $A \in \mathbb{F}^{n\times n}$ where $\mathbb{F}$ is field and n is a positive integer.
TRUE / POSSIBLE / FALSE: A can be written as a finite product of elementary matrices.

(c) Let $W = \{ax^2 + bx + c \mid; a + b = 0, c = 1\}$
TRUE / POSSIBLE / FALSE: W is a subspace of $\mathbb{R}[x]$.

(d) Let $A \in \mathbb{C}^{n\times n}$ (for some positive integer $n > 1$) and suppose $\det(A) = 3$. Let E be the $n \times n$ elementary matrix associated with the row operation: add 5 times row 2 to row 1.
TRUE / POSSIBLE / FALSE: $\det(-E^T A^{-2}) = -\dfrac{1}{9}$.

(e) Let $A \in \mathbb{R}^{3\times 3}$ have characteristic polynomial $f(t) = t^3 - 2t^2 + t$. In addition suppose that the eigenspace for eigenvalue $\lambda = 1$ is $E_1 = \left\{ \begin{bmatrix} a \\ b \\ -a \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$.
TRUE / POSSIBLE / FALSE: A is diagonalizable.

(f) Let n be a positive integer.
TRUE / POSSIBLE / FALSE: $\mathcal{L}(\mathbb{R}^3, \mathbb{R}^{n+1}) \cong \mathbb{R}^{n\times n} \oplus P_{2n}$.

(g) Let $T : P_5 \to P_5$ be a one-to-one linear transformation.
TRUE / POSSIBLE / FALSE: T is onto.

(h) Suppose that $A \in \mathbb{R}^{n\times n}$ and that $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b} \in \mathbb{R}^{n\times 1}$
TRUE / POSSIBLE / FALSE: $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

(i) Let $T : \mathbb{R}^{2\times 2} \to P_5$ be a linear transformation. Assume that $\mathrm{rank}(T) = 2$.
TRUE / POSSIBLE / FALSE: $\mathrm{Ker}(T) \cong T(\mathbb{R}^{2\times 2})$ (i.e. the kernel and range of T are isomorphic).

(j) Let $A \in \mathbb{Z}^{3\times 3}$ (i.e. A is a 3×3 matrix with integer entries).
TRUE / POSSIBLE / FALSE: $\det(A^{-1}) = \sqrt{5}$.

(k) Let $T : P_2 \to \mathbb{R}^{2\times 3}$ be a linear transformation.
TRUE / POSSIBLE / FALSE: $\mathrm{rank}(T) = 4$.

(l) Let $S \subset V$ where V is an inner product space (over some field $\mathbb{F}$). Suppose that S is an orthogonal set.
TRUE / POSSIBLE / FALSE: S is linearly independent.

(m) Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix.
TRUE / POSSIBLE / FALSE: $\det(A) \in \mathbb{R}$.

(n) Let $A \in \mathbb{R}^{3\times 3}$ have characteristic polynomial $f(t) = t^3 - 2t^2 - t$
TRUE / POSSIBLE / FALSE: A is invertible.

(o) Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a one-to-one linear transformation.
TRUE / POSSIBLE / FALSE: T is onto.

(p) Let $T : P_3 \to \mathbb{R}^5$ be a linear transformation.
TRUE / POSSIBLE / FALSE: T is one-to-one.

(q) Let $T : P_3 \to \mathbb{R}^5$ be a linear transformation.
TRUE / POSSIBLE / FALSE: T is onto.

(r) Recall that a square matrix P is called a permutation matrix if the rows of P are just a permutation of the rows of the identity matrix. Let P be a permutation matrix.
TRUE / POSSIBLE / FALSE: P is orthogonal.

(s) Let $Q \in \mathbb{R}^{n\times n}$ be an orthogonal matrix.
TRUE / POSSIBLE / FALSE: $-Q$ is orthogonal.