Multivariate Limits are Weird

Here we present a function $f(x, y) = \frac{xy^2}{x^2 + y^4}$ such that the limit approaching the origin, $(0, 0)$, along any line is 0.
However, the limit approaching the origin along the parabola, $x = y^2$, is 1/2. Thus approaching a point from "every direction"
(i.e., along every possible line) is not enough. To establish a multivariate limit exists one must approach along every possible
curve. This means that trying to establish a limit using approaches along curves isn't practical.

$$f(x, y) = \frac{xy^2}{y^4 + x^2}$$

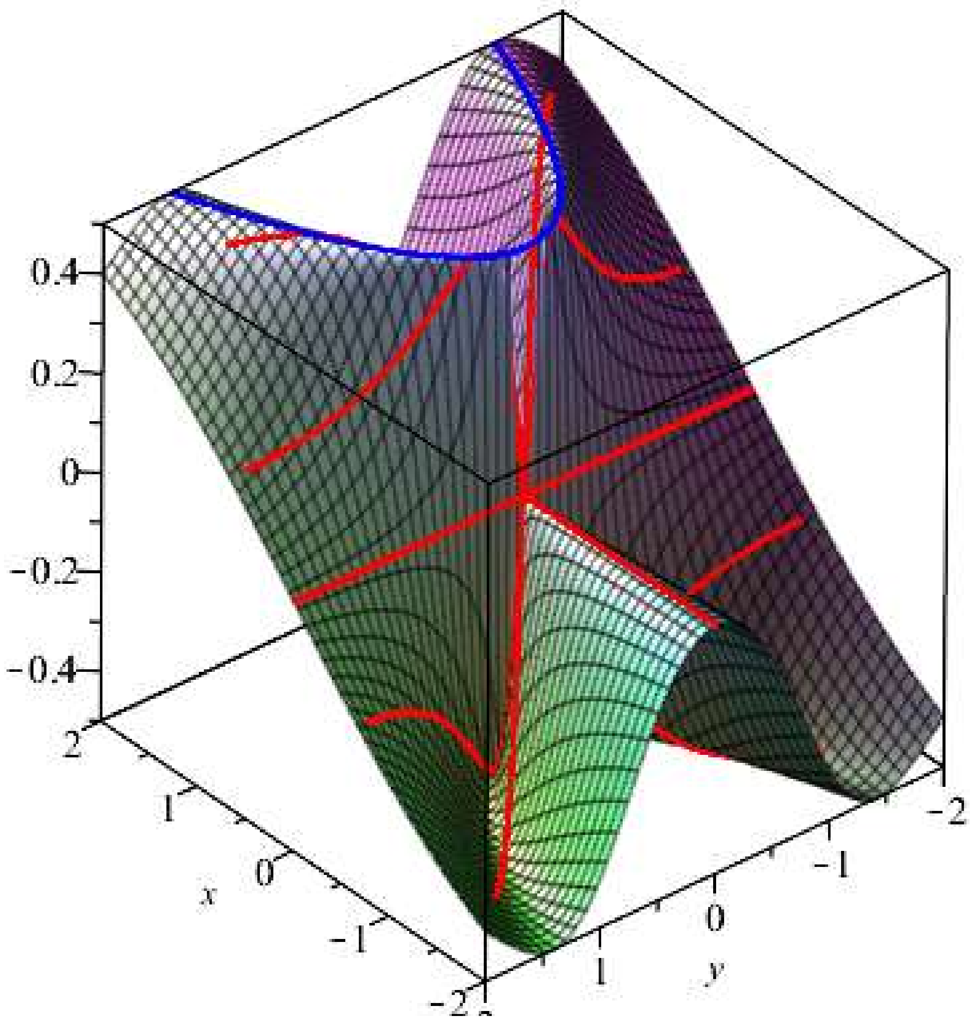
$$0, 0$$

$$x = y^2 \tag{1}$$

```
> restart;
  with(plots):
> f := (x,y) -> x*y^2/(x^2+y^4):
  'f(x,y)' = f(x,y);
```

$$f(x, y) = \frac{x\,y^2}{y^4 + x^2} \tag{2}$$

```
> surfPlot := plot3d(f(x,y),x=-2..2,y=-2..2,numpoints=50000):
  linePlots := [seq(spacecurve(<x/sqrt(1+m^2),m*x/sqrt(1+m^2),f
  (x/sqrt(1+m^2),m*x/sqrt(1+m^2))>,
                              x=-2..2,color=red,thickness=2,
  numpoints=5000),m in {-3,-1,0,1,3}),
                spacecurve(<0,y,f(0,y)>,y=-2..2,color=red,
  thickness=2,numpoints=5000)]:
  parabolaPlot := spacecurve(<y^2,y,f(y^2,y)>,y=-sqrt(2)..sqrt(2),
  color=blue,thickness=2,numpoints=5000):
> display(surfPlot,linePlots,parabolaPlot);
```



Notice that the surface heads to $z = 0$ along every trace made by a vertical plane (i.e. heading into the origin along straight lines in the domain). However, if we approach the origin along the parabola $x = y^2$ we get z approaching $\frac{1}{2}$. Thus the limit does not exist.

The red lines are the paths traced out on the surface following lines heading into the origin in the domain. The blue parabola is the path traced out on the surface if we follow the parabolic path in the domain. Notice that the blue parabola kind of jumps over a canyon. The graph doesn't actually have a gap there, Maple just has a difficult time plotting near the origin where we have a "rip" in the surface.