An n-dimensional Pythagorean theorem

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What should the Pythagorean theorem look like in higher dimensions? One might claim that it is the same in all Euclidean spaces because the sum of the squares of the legs of a right triangle equals the square of the length of the hypotenuse regardless of dimension. This is not terribly exciting. However, if we shift our perspective a bit, we are led to more interesting answer.

First, we recast the Pythagorean theorem in terms of projections. Let a vector $\mathbf{v} = \langle a, b \rangle = a\mathbf{i} + b\mathbf{j}$ in $\mathbb{R}^2$ be projected onto the x and y-axes: $\mathrm{proj}_{\mathbf{i}}\mathbf{v} = \langle a, 0 \rangle = a\mathbf{i}$ and $\mathrm{proj}_{\mathbf{j}}\mathbf{v} = \langle 0, b \rangle = b\mathbf{j}$.

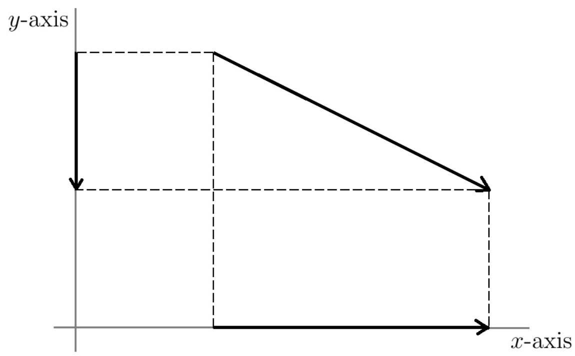

Figure 1: The Pythagorean theorem in 2 dimensions.

The Pythagorean theorem tells us that

$$\|\mathbf{v}\|^2 = a^2 + b^2 = \|\mathrm{proj}_{\mathbf{i}}\mathbf{v}\|^2 + \|\mathrm{proj}_{\mathbf{j}}\mathbf{v}\|^2, \tag{1}$$

where the length of $\mathbf{v}$ is $\|\mathbf{v}\| = \sqrt{\mathbf{v} \bullet \mathbf{v}}$ and $\mathbf{v} \bullet \mathbf{w}$ is the dot product of $\mathbf{v}$ and $\mathbf{w}$. This suggests the picture in Figure 2.

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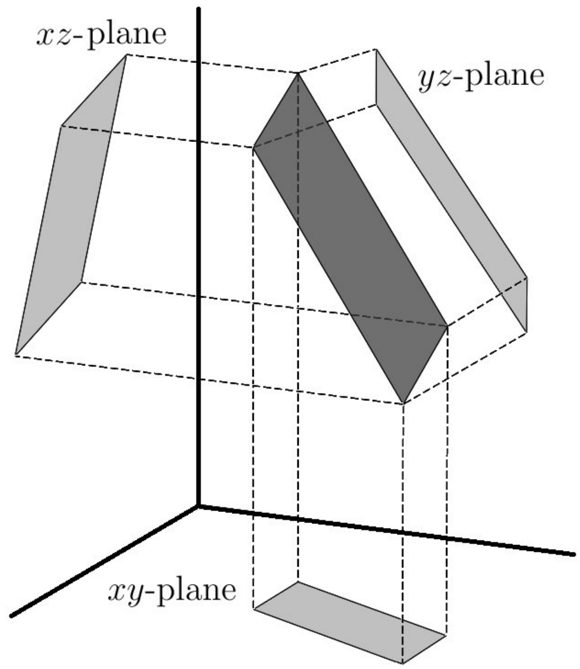

Figure 2: The Pythagorean theorem in 3 dimensions.

Our 3-dimensional Pythagorean theorem asserts that the square of the area of a parallelogram is equal to the sum of the squares of the areas of its projections onto the coordinate planes. This follows immediately from:

$$\left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \right\|^2 = \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & v_2 & v_3 \\ 0 & w_2 & w_3 \end{vmatrix} \right\|^2 + \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & 0 & v_3 \\ w_1 & 0 & w_3 \end{vmatrix} \right\|^2 + \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & 0 \\ w_1 & w_2 & 0 \end{vmatrix} \right\|^2,$$

where the $\mathbf{v}$ and $\mathbf{w}$ are the sides of the parallelogram. This restating of (1), suggests

$$\left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ a & b \end{vmatrix} \right\|^2 = \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ 0 & b \end{vmatrix} \right\|^2 + \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ a & 0 \end{vmatrix} \right\|^2.$$

These identities are special cases of a more general identity. Before stating and proving our general proposition, we establish notation and gather a few facts. Mainly we require a generalized cross product in n-dimensions.

Cross Products

Let $\mathbf{e}_i$ be the i^{th} standard, unit vector in $\mathbb{R}^n$. Notice that the dot product in effect replaces the unit vectors in the first factor of the product by the corresponding components of the second factor. For example,

$$(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \bullet (4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}) = (\boxed{4}) + 2(\boxed{5}) + 3(\boxed{6}),$$

so $\mathbf{i}$ is replaced by 4, $\mathbf{j}$ by 5, and $\mathbf{k}$ by 6.

The *generalized cross product* in n-dimensions is the determinant

$$\mathbf{w}_1 \times \mathbf{w}_2 \times \cdots \times \mathbf{w}_{n-1} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ w_{11} & w_{12} & \cdots & w_{1n} \\ \vdots & \vdots & & \vdots \\ w_{n-1\,1} & w_{n-1\,2} & \cdots & w_{n-1\,n} \end{vmatrix}, \tag{2}$$

where $\mathbf{w}_i = \langle w_{i1}, w_{i2}, \ldots, w_{in} \rangle$. As in three dimensions, we calculate the generalized cross product by expanding (2) by minors:

$$\mathbf{w}_1 \times \mathbf{w}_2 \times \cdots \times \mathbf{w}_{n-1} = M_1\mathbf{e}_1 - M_2\mathbf{e}_2 + \cdots + (-1)^{n+1} M_n \mathbf{e}_n \in \mathbb{R}^n, \tag{3}$$

where

$$M_i(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-1}) = \begin{vmatrix} w_{11} & \cdots & w_{1\,i-1} & w_{1\,i+1} & \cdots & w_{1n} \\ w_{21} & \cdots & w_{2\,i-1} & w_{2\,i+1} & \cdots & w_{2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ w_{n-1\,1} & \cdots & w_{n-1\,i-1} & w_{n-1\,i+1} & \cdots & w_{n-1\,n} \end{vmatrix}. \tag{4}$$

Since the dot product replaces the standard unit vectors with the corresponding components of the vector being dotted, for $\mathbf{v} = \langle v_1, \ldots, v_n \rangle \in \mathbb{R}^n$ we have

$$(\mathbf{w}_1 \times \mathbf{w}_2 \times \cdots \times \mathbf{w}_{n-1}) \bullet \mathbf{v} = \begin{vmatrix} v_1 & v_2 & \cdots & v_n \\ w_{11} & w_{12} & \cdots & w_{1n} \\ \vdots & \vdots & & \vdots \\ w_{n-1\,1} & w_{n-1\,2} & \cdots & w_{n-1\,n} \end{vmatrix}.$$

The most important property of the cross product is that it produces a vector orthogonal to its inputs. Our generalized cross product has this property. For any $i = 1, \ldots, n-1$, $(\mathbf{w}_1 \times \mathbf{w}_2 \times \cdots \times \mathbf{w}_{n-1}) \bullet \mathbf{w}_i = 0$ since the row $\mathbf{w}_i$ would appear twice in the determinant and a determinant of a matrix with a repeated row is 0.

We should mention that this generalization of the cross product is not new. Readers familiar with exterior algebra (see [1]) will notice that

$$\mathbf{w}_1 \times \mathbf{w}_2 \times \cdots \times \mathbf{w}_{n-1} = *(\mathbf{w}_1 \wedge \mathbf{w}_2 \wedge \cdots \wedge \mathbf{w}_{n-1}),$$

where $*$ is the Hodge dual operator. We also point the interested reader to Massey's very accessible paper [2] which classifies all possible cross products in a very general setting.

Pythagorean Theorem

Our next task is to connect the cross product with geometry. It is well known that determinants yield signed volumes. Specifically, the area of a parallelogram spanned by $\mathbf{v} = \langle v_1, v_2 \rangle, \mathbf{w} = \langle w_1, w_2 \rangle \in \mathbb{R}^2$ is given by the absolute value of $\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = v_1w_2 - v_2w_1$. Likewise, the absolute value of $\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ is the volume of the parallelepiped spanned by $\mathbf{u} = \langle u_1, u_2, u_3 \rangle, \mathbf{v} = \langle v_1, v_2, v_3 \rangle, \mathbf{w} = \langle w_1, w_2, w_3 \rangle \in \mathbb{R}^3$. In general, the absolute value of the determinant of an $n \times n$ matrix can be interpreted as the n-dimensional volume of the *parallelotope* spanned by the rows of that matrix. The sign (which we obliterate by taking absolute values) tracks the parallelotope's orientation, which in three dimensions is right or left handedness.

Now let $\mathbf{v} = \mathbf{w}_1 \times \mathbf{w}_2 \times \cdots \times \mathbf{w}_{n-1}$ for $\mathbf{w}_i \in \mathbb{R}^n$. If $\mathbf{v} = \mathbf{0}$, then the $\mathbf{w}_i$'s are linearly dependent (and their $(n-1)$-dimensional volume is zero). Alternatively, if $\mathbf{v} \neq \mathbf{0}$, $\|\mathbf{v}\| = \mathbf{v} \bullet \frac{\mathbf{v}}{\|\mathbf{v}\|}$. Now $\mathbf{v}$ is orthogonal to each $\mathbf{w}_i$, and so is orthogonal to the hyperplane spanned by the $\mathbf{w}_i$'s. Thus the $n \times n$ determinant $\|\mathbf{v}\| = (\mathbf{w}_1 \times \mathbf{w}_2 \times \cdots \times \mathbf{w}_{n-1}) \bullet \frac{\mathbf{v}}{\|\mathbf{v}\|}$ is the n-dimensional volume of the parallelotope spanned by the $\mathbf{w}_i$'s plus $\frac{\mathbf{v}}{\|\mathbf{v}\|}$. But $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is orthogonal to the $\mathbf{w}_i$'s and of length 1, so the n-dimensional volume of the whole parallelotope equals the $(n-1)$-dimensional volume of the parallelotope spanned by the $\mathbf{w}_i$'s, using the claim: "base times height = volume" and sweeping the actual geometry under the carpet. In summary, $\|\mathbf{w}_1 \times \mathbf{w}_2 \times \cdots \times \mathbf{w}_{n-1}\|$ is the $(n-1)$-dimensional volume of the parallelotope spanned by the $\mathbf{w}_i$'s. Now we are ready for our n-dimensional Pythagorean theorem.

Theorem *The square of the $(n-1)$-dimensional volume of the parallelotope spanned by $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-1} \in \mathbb{R}^n$ equals the sum of the squares of the $(n-1)$-dimensional volumes of the projections of this parallelotope onto the coordinate hyperplanes.*

Proof: Our cross product $\mathbf{v} = \mathbf{w}_1 \times \mathbf{w}_2 \times \cdots \times \mathbf{w}_{n-1}$ is a vector whose length, as we have just argued, is the $(n-1)$-dimensional volume of the parallelotope spanned by the $\mathbf{w}_i$'s. But the component M_i of $\mathbf{v}$ (see (3)) is (in absolute value) the $(n-1)$-dimensional volume of the projection of this parallelotope onto the coordinate hyperplane spanned by $\mathbf{w}_1, \ldots, \mathbf{w}_{i-1}, \mathbf{w}_{i+1}, \ldots, \mathbf{w}_{n-1}$ (see (4)). In summary, taking the norm-squared of (3) yields our theorem. ■

Summary

An n-dimensional generalization of the standard cross product, leads to an n-dimensional generalization of the Pythagorean theorem.

References

[1] R. W. R. Darling, *Differential Forms and Connections*, Cambridge University Press, Cambridge UK, 1994.

[2] W. S. Massey, Cross products of vectors in higher dimensional Euclidean spaces, *Amer. Math. Monthly* **90** (1983) 697–701.