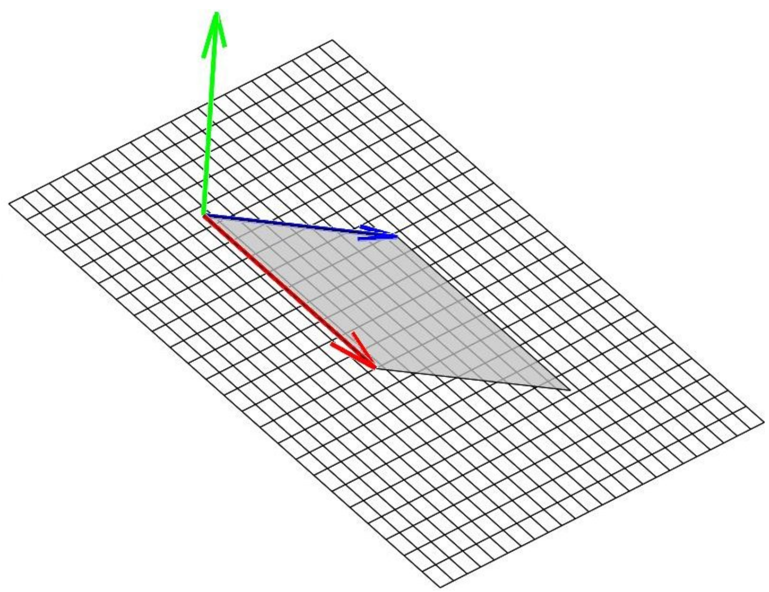
Cross Products

&

An n-dimensional Pythagorean Theorem

The Cross Product



Given $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, the cross product of $\mathbf{v}$ and $\mathbf{w}$ is

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k} \\ &= \Big\langle v_2 w_3 - v_3 w_2,\ -(v_1 w_3 - v_3 w_1),\ v_1 w_2 - v_2 w_1 \Big\rangle \end{aligned}$$

$\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ are the standard unit vectors.

The Cross Product – Geometrically

The cross product of $\mathbf{v}$ and $\mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. This leaves a whole 1-dimensional subspace of possibilities.

Next, the cross product has the same length as the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$. Now we are left with at most two options.

Finally, the cross product obeys the "right hand rule". This now completely determines the cross product.

Can this be done in $\mathbb{R}^n$? In $\mathbb{R}^3$, we have $3 - 2 = 1$.

In $\mathbb{R}^n$, we have $n - ??? = 1$.

A Generalized Cross Product

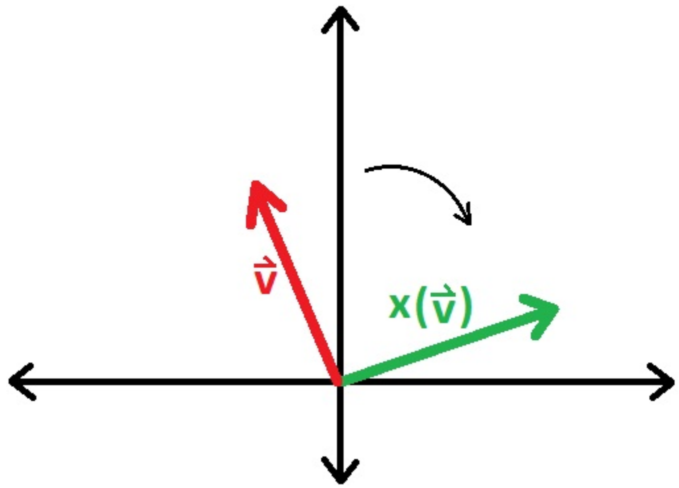
It turns out that the cross product can be generalized to n-dimensional vectors. Keeping in mind that the cross product is specifying a direction orthogonal (i.e. perpendicular) to given vectors, we need to specify $n-1$ directions in n-dimensional space so only 1 direction is left. So why does the cross product have 2 inputs in 3 dimensions? It's as simple as $2+1=3$.

Let $\mathbf{w}_1 = \langle w_{11}, w_{12}, \ldots, w_{1n} \rangle$, $\mathbf{w}_2 = \langle w_{21}, w_{22}, \ldots, w_{2n} \rangle$, ..., $\mathbf{w}_{n-1} = \langle w_{n-1\,1}, w_{n-1\,2}, \ldots, w_{n-1\,n} \rangle$ be our $n-1$ input vectors in $\mathbb{R}^n$ (n-dimensional space). Then we can form a sort of cross product (which matches the regular cross product in 3-dimensions) as follows:

$$\mathbf{w}_1 \times \mathbf{w}_2 \times \cdots \times \mathbf{w}_{n-1} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & & \vdots \\ w_{n-1\,1} & w_{n-1\,2} & \cdots & w_{n-1\,n} \end{vmatrix}$$

Here, $\mathbf{e}_1 = \langle 1, 0, 0, \ldots, 0 \rangle$, $\mathbf{e}_2 = \langle 0, 1, 0, \ldots, 0 \rangle$ etc. are the standard unit vectors.

The Cross Product in 2D



$$\times(\langle v_1, v_2\rangle) = \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ v_1 & v_2 \end{vmatrix} = \langle v_2, -v_1\rangle$$

Notice that $\langle v_1, v_2\rangle \bullet \langle v_2, -v_1\rangle = 0$ and $\|\langle v_1, v_2\rangle\| = \|\langle v_2, -v_1\rangle\|$.

The Dot Product: An Odd Perspective

Computing a dot product is easy:

$$\langle 1, 2, 3 \rangle \bullet \langle 4, 5, 6 \rangle = 1(4) + 2(5) + 3(6) = 32$$

We simply multiply the corresponding components and then sum the resulting products.

Here's a second way of thinking about this computation: Rewrite our vectors in terms of standard unit vectors: $1\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$. To compute the dot product, simply replace each unit vector with the corresponding component of the second vector:

$$(1\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \bullet (4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}) = 1\boxed{4} + 2\boxed{5} + 3\boxed{6} = 32$$

The Dot Product: An Odd Perspective

So dotting $\mathbf{w}_1 \times \mathbf{w}_2 \times \cdots \times \mathbf{w}_{n-1}$ with a vector $\mathbf{w}_n = \langle w_{n1}, w_{n2}, \ldots, w_{nn} \rangle$ amounts to replacing the standard unit vectors along the top row of the determinant defining the cross product with the components of $\mathbf{w}_n$.

$$\mathbf{w}_n \bullet (\mathbf{w}_1 \times \mathbf{w}_2 \times \cdots \times \mathbf{w}_{n-1}) = \begin{vmatrix} w_{n1} & w_{n2} & \cdots & w_{nn} \\ w_{11} & w_{12} & \cdots & w_{1n} \\ \vdots & \vdots & & \vdots \\ w_{n-1\,1} & w_{n-1\,2} & \cdots & w_{n-1\,n} \end{vmatrix}$$

From multivariable calculus we might recognize the triple scalar product:

$$\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Determinant = Signed Volume

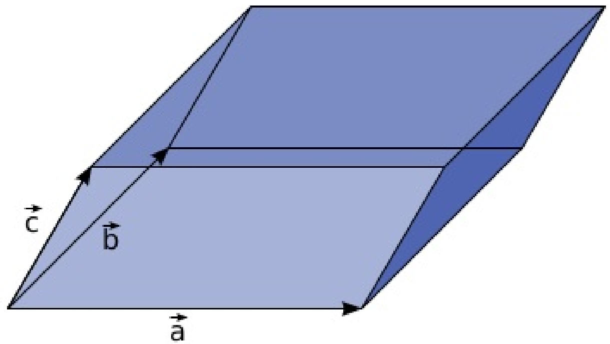
Notice $\mathbf{w}_1 \times \mathbf{w}_2 \times \cdots \times \mathbf{w}_{n-1}$ is orthogonal to each of the vectors $\mathbf{w}_1$, $\mathbf{w}_2$, . . . , $\mathbf{w}_{n-1}$ since the determinant of a matrix with a repeated row is zero.

Next, recall [in the graduate school sense] from your introduction to linear course that determinants compute volumes. In particular, the determinant of a 2×2 matrix:

$$\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = v_1 w_2 - w_1 v_2$$

is the area of the parallelogram spanned by $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$...well almost. It **is** the area if $\mathbf{v}$ and $\mathbf{w}$ are oriented in a counterclockwise manner and negative the area if oriented in a clockwise manner or zero if the vectors are parallel.

Determinant = Signed Volume



The parallelepiped spanned by the vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$

In the same manner a 3×3 determinant computes $\pm$ the volume of the parallelepiped spanned by its rows ($+$ if right-handed, $-$ if left handed, and 0 if coplanar).

In general n vectors span a parallelotope and the corresponding determinant computes $\pm$ its n-dimensional volume.

What is $\|\mathbf{w}_1 \times \mathbf{w}_2 \times \cdots \times \mathbf{w}_{n-1}\|$?

Let $\mathbf{v} = \mathbf{w}_1 \times \cdots \times \mathbf{w}_{n-1}$.

Then $\|\mathbf{v}\| = \dfrac{\|\mathbf{v}\|^2}{\|\mathbf{v}\|} = \dfrac{\mathbf{v} \bullet \mathbf{v}}{\|\mathbf{v}\|} = \left(\dfrac{\mathbf{v}}{\|\mathbf{v}\|}\right) \bullet \mathbf{v}$ is an $n \times n$ determinant, so it computes the (n-dimensional) volume of the parallelotope spanned by $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-1}$, and $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$.

For a parallelotope, "volume = base × height". But $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector orthogonal to $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-1}$. Thus the volume is equal to the (($n-1$)-volume) of the *base* (the parallelotope spanned by $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-1}$) times the *height* $= 1$.

This means that $\|\mathbf{w}_1 \times \cdots \times \mathbf{w}_{n-1}\|$ is the $(n-1)$-dimensional volume of the parallelotope spanned by $\mathbf{w}_1, \ldots, \mathbf{w}_{n-1}$.

Characterizing the Generalized Cross Product

If $\mathbf{v} = \mathbf{w}_1 \times \cdots \times \mathbf{w}_{n-1}$, we find that the vectors $\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-1}$ are *positively oriented* in some sense (generalizing *counterclockwise* in $\mathbb{R}^2$ and *right handed* in $\mathbb{R}^3$).

Thus our generalized cross product is completely determined by...

- being orthogonal to its inputs
- having its length is equal to the volume of the parallelotope spanned by its inputs
- completing the list of inputs to a positively oriented basis

[*Fine Print:* Unless the inputs are linearly dependent.
In such a case, we just get the zero vector.]

Note: This cross product is compatible with rotations. Specifically, if R is a rotation, then

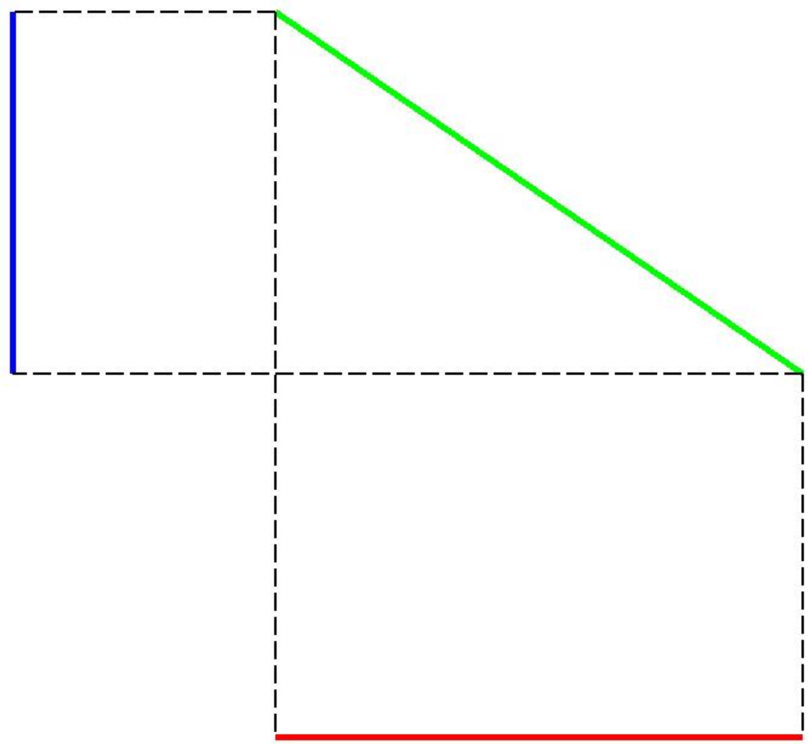
$$R(\mathbf{w}_1) \times R(\mathbf{w}_2) \times \cdots \times R(\mathbf{w}_{n-1}) = R(\mathbf{w}_1 \times \mathbf{w}_2 \times \cdots \times \mathbf{w}_{n-1}).$$

Application:

An n-Dimensional Pythagorean Theorem

The 2D-Theorem Revisited

Projections in 2 dimensions. . .



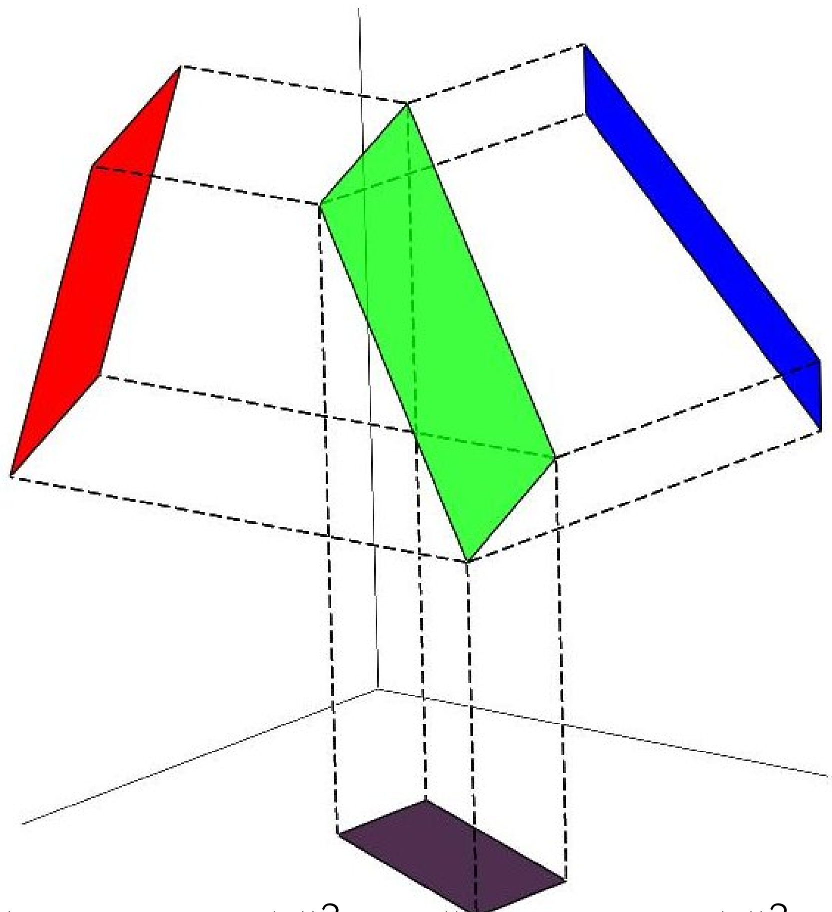
Instead of viewing the classical Pythagorean theorem as a statement about right triangles, we can view it as a statement about a vector and its projections (think shadows) on the x- and y-axes.

The theorem now says that the square of the length of a vector is equal to the sum of the squares of the lengths of its projections.

$$\|\langle a, b\rangle\|^2 = \|\langle 0, b\rangle\|^2 + \|\langle a, 0\rangle\|^2$$

The Pythagorean Theorem in $3D$

Projections in 3 dimensions...



For the 3D version of the theorem we should use a parallelogram (a 2-dimensional object) instead of a vector (a 1-dimensional object). Now instead of a statement about lengths, we have a statement about areas.

The $3D$ version of the theorem says that the sum of the squares of the areas of the projections of a parallelogram is equal to the sum of the square of the area of that parallelogram.

$$\underbrace{\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\right\|^2}_{\text{Original Parallelogram}} = \underbrace{\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & v_2 & v_3 \\ 0 & w_2 & w_3 \end{vmatrix}\right\|^2}_{yz\text{-Plane Projection}} + \underbrace{\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & 0 & v_3 \\ w_1 & 0 & w_3 \end{vmatrix}\right\|^2}_{xz\text{-Plane Projection}} + \underbrace{\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & 0 \\ w_1 & w_2 & 0 \end{vmatrix}\right\|^2}_{xy\text{-Plane Projection}}$$

The Pythagorean Theorem in n-Dimensions

The n-dimensional Pythagorean theorem says that the square of the n-dimensional volume of a $(n-1)$-dimensional parallelotope in n-dimensional space is equal to the sum of the squares of the n-dimensional volumes of the projections of this parallelotope onto each of the coordinates planes in n-dimensional space.

This theorem follows quite easily from basic vector and determinant properties. Here is the statement of the theorem in terms of generalized cross products:

$$\left\| \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ w_{11} & w_{12} & \cdots & w_{1n} \\ \vdots & \vdots & & \vdots \\ w_{n-1\,1} & w_{n-1\,2} & \cdots & w_{n-1\,n} \end{vmatrix} \right\|^2 =$$

$$\left\| \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ 0 & w_{12} & \cdots & w_{1n} \\ \vdots & \vdots & & \vdots \\ 0 & w_{n-1\,2} & \cdots & w_{n-1\,n} \end{vmatrix} \right\|^2 + \left\| \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ w_{11} & 0 & \cdots & w_{1n} \\ \vdots & \vdots & & \vdots \\ w_{n-1\,1} & 0 & \cdots & w_{n-1\,n} \end{vmatrix} \right\|^2 + \cdots + \left\| \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ w_{11} & w_{12} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ w_{n-1\,1} & w_{n-1\,2} & \cdots & 0 \end{vmatrix} \right\|^2$$

In this notation, the classical (i.e. 2D) Pythagorean theorem looks like:

$$\left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ a & b \end{vmatrix} \right\|^2 = \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ 0 & b \end{vmatrix} \right\|^2 + \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ a & 0 \end{vmatrix} \right\|^2$$

Strange Algebras and Other Cross Products

Exterior Algebras

Let V be an n-dimensional vector space (over $\mathbb{R}$). Let $\wedge$ be an operation on vectors such that...

- This product is bilinear: $(\mathbf{v}+\mathbf{w})\wedge\mathbf{u} = \mathbf{v}\wedge\mathbf{u}+\mathbf{w}\wedge\mathbf{u}$, $\mathbf{v}\wedge(\mathbf{u}+\mathbf{w}) = \mathbf{v}\wedge\mathbf{u}+\mathbf{v}\wedge\mathbf{w}$, and $(c\mathbf{v})\wedge\mathbf{w} = c(\mathbf{v}\wedge\mathbf{w}) = \mathbf{v}\wedge(\mathbf{w})$.
- This product is alternating (equivalently skew-symmetric): $\mathbf{v}\wedge\mathbf{v}=\mathbf{0}$ (equivalently $\mathbf{v}\wedge\mathbf{w} = -\mathbf{w}\wedge\mathbf{v}$).

We get $\wedge V = \wedge^0 V \oplus \wedge^1 V \oplus \cdots \wedge^n V = \mathbb{R}\oplus V \oplus \cdots$ which is called the exterior algebra (or Grassmann algebra) over V.

Example: Consider $V=\mathbb{R}^3$. Then $\wedge^0 V = \mathbb{R}$, $\wedge^1 V = \text{span}\{\mathbf{i},\mathbf{j},\mathbf{k}\}$, $\wedge^2 V = \text{span}\{\mathbf{i}\wedge\mathbf{j}, \mathbf{j}\wedge\mathbf{k}, \mathbf{k}\wedge\mathbf{i}\}$, and $\wedge^3 V = \text{span}\{\mathbf{i}\wedge\mathbf{j}\wedge\mathbf{k}\}$.

$(\mathbf{i}+2\mathbf{k})\wedge(2\mathbf{i}-\mathbf{j}+3\mathbf{k}) = \mathbf{i}\wedge(2\mathbf{i}-\mathbf{j}+3\mathbf{k}) + 2\mathbf{k}\wedge(2\mathbf{i}-\mathbf{j}+3\mathbf{k}) = 2\mathbf{i}\wedge\mathbf{i}-\mathbf{i}\wedge\mathbf{j}+3\mathbf{i}\wedge\mathbf{k}+4\mathbf{k}\wedge\mathbf{i}-2\mathbf{k}\wedge\mathbf{j}+6\mathbf{k}\wedge\mathbf{k} = -\mathbf{i}\wedge\mathbf{j}-3\mathbf{k}\wedge\mathbf{i}+4\mathbf{k}\wedge\mathbf{i}+2\mathbf{j}\wedge\mathbf{k} = -\mathbf{i}\wedge\mathbf{j}+2\mathbf{j}\wedge\mathbf{k}+\mathbf{k}\wedge\mathbf{i}$.

Exterior Algebras: Hodge Dual

It turns out that the dimension of the k^{th}-piece of the exterior algebra is $\dim(\wedge^k V) = \binom{n}{k} = \dfrac{n!}{k!(n-k)!}$. Notice that $\dim(\wedge^k V) = \binom{n}{k} = \binom{n}{n-k} = \dim(\wedge^{n-k} V)$. This points towards the following operation:

The Hodge Dual is an isomorphism, $\star$, between $\wedge^k V$ and $\wedge^{n-k} V$. For example: Given the standard unit vector basis for $\mathbb{R}^n$, we replace $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_k$ with $\mathbf{e}_{k+1} \wedge \mathbf{e}_{k+2} \wedge \cdots \wedge \mathbf{e}_n$.

Example: Consider $V = \mathbb{R}^3$. Then $\star(\mathbf{i}) = \mathbf{j} \wedge \mathbf{k}$, $\star(\mathbf{j}) = \mathbf{k} \wedge \mathbf{i}$, and $\star(\mathbf{k}) = \mathbf{i} \wedge \mathbf{j}$. Also, $\star(\mathbf{i} \wedge \mathbf{j}) = \mathbf{k}$ etc.

$\star(-\mathbf{i} \wedge \mathbf{j} + 2\mathbf{j} \wedge \mathbf{k} + \mathbf{k} \wedge \mathbf{i}) = -\mathbf{k} + 2\mathbf{i} + \mathbf{j} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

Notice that $(\mathbf{i} + 2\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$. Coincidence?

Exterior Algebras & The Cross Product

Let $\mathbf{w}_1, \ldots, \mathbf{w}_{n-1} \in \mathbb{R}^n$ (assuming $n > 2$). Then

$$\mathbf{w}_1 \times \mathbf{w}_2 \times \cdots \times \mathbf{w}_{n-1} = \star(\mathbf{w}_1 \wedge \mathbf{w}_2 \wedge \cdots \wedge \mathbf{w}_{n-1})$$

In particular, if $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, then

$$\mathbf{v} \times \mathbf{w} = \star(\mathbf{v} \wedge \mathbf{w})$$

Binary Cross Products

Let $\times : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a bilinear product such that

- For all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.
- For all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $\|\mathbf{v} \times \mathbf{w}\|^2 = \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 - (\mathbf{v} \bullet \mathbf{w})^2$

then $n = 3$ or 7.

Why?

Consider $(a, \mathbf{v}), (b, \mathbf{w}) \in \mathbb{R} \oplus \mathbb{R}^n = \mathcal{A}$. If such a cross product exists, we can define

$$(a, \mathbf{v})(b, \mathbf{w}) = (ab - \mathbf{v} \bullet \mathbf{w}, a\mathbf{w} + b\mathbf{v} + \mathbf{v} \times \mathbf{w}).$$

It is easy to see that this product is bilinear and has a two-sided multiplicative unit $(1, \mathbf{0})$. Also, using the properties of $\times$ above, one can show that

$$\|(a, \mathbf{v})(b, \mathbf{w})\| = \|(a, \mathbf{v})\| \|(b, \mathbf{w})\|$$

which means that $\mathcal{A}$ is a normed division algebra.

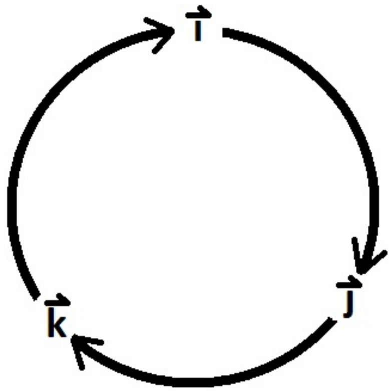
But normed division algebras are very rare.

Normed Division Algebras

Theorem [Hurwitz]: (Up to isomorphism) The only normed division algebras are: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (the quaternions), and $\mathbb{O}$ (the octonions). These algebras are distinguished by: $\mathbb{R}$ is an ordered field, $\mathbb{C}$ is algebraically closed, $\mathbb{H}$ is associative but not commutative, and $\mathbb{O}$ is not associative.

A power associative algebra is an algebra such that every 1-dimensional subalgebra is associative. An alternative algebra is an algebra such that every 2-dimensional subalgebra is associative. The word "normed" can be swapped with the word "alternative" in the theorem above [this version is due to Zorn].

Quaternions



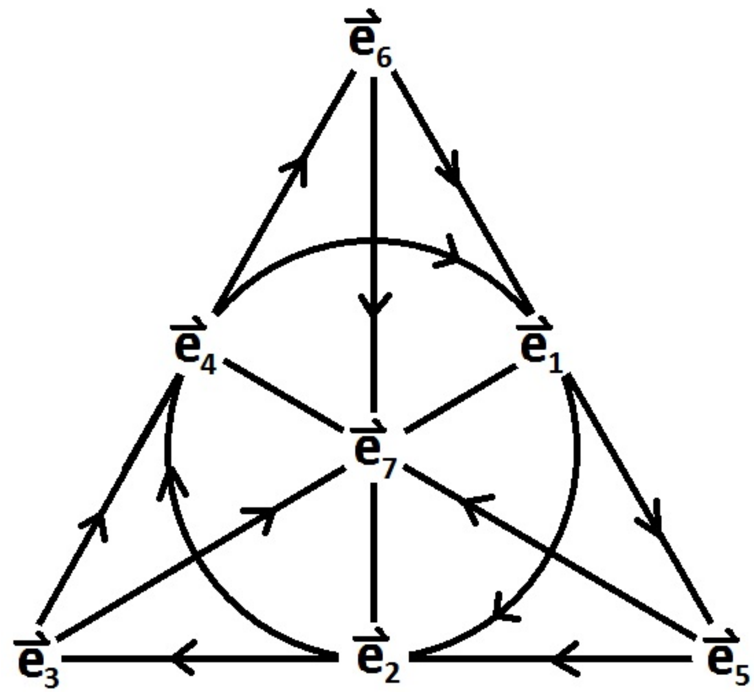
"Circle of Doom"

$\mathbb{H} = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbb{R}\}$.

$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = \mathbf{k}$, $\mathbf{ji} = -\mathbf{k}$ etc.

For pure imaginary quaternions: $\mathbf{v}, \mathbf{w}$ we have $\mathbf{vw} = \mathbf{v} \times \mathbf{w} - \mathbf{v} \bullet \mathbf{w}$.

Octonions



$$\mathbb{O} = \{a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_7\mathbf{e}_7 \mid a_0, a_1, \ldots, a_7 \in \mathbb{R}\}$$

$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \cdots = \mathbf{e}_7^2 = -1$ and $\mathbf{e}_1\mathbf{e}_5 = \mathbf{e}_6$, $\mathbf{e}_3\mathbf{e}_7 = \mathbf{e}_1$, etc.

Pure imaginary octonions correspond to vectors in $\mathbf{v}, \mathbf{w} \in \mathbb{R}^7$.
Then $\mathbf{vw} = \mathbf{v} \times \mathbf{w} - \mathbf{v} \bullet \mathbf{w}$.

Binary Cross Products

The previous theorem about binary cross products can be weakened...

Let $\times : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous product such that

- For all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.
- For all non-parallel vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $\mathbf{v} \times \mathbf{w} \neq \mathbf{0}$

then $n = 3$ or 7.

Moreover, if in addition, whenever R is a rotation, we have $R(\mathbf{v} \times \mathbf{w}) = R(\mathbf{v}) \times R(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, then $n = 3$.

So in the end, maybe there really is only one cross product... Hmmm...

Other Cross Products?

Beno Eckmann proposed the following definition:

An r-airy product $\times$ on $\mathbb{R}^n$ is a vector cross product if $\times$ is continuous, the product is orthgonal to its inputs: $(\mathbf{w}_1 \times \cdots \times \mathbf{w}_r) \bullet \mathbf{w}_j = 0$ for each j, and $\|\mathbf{w}_1 \times \cdots \times \mathbf{w}_r\|^2 = \det(\mathbf{w}_i \bullet \mathbf{w}_j)$.

Eckmann and Whitehead then were able to establish that vector cross products only exists if...

- n is even and $r = 1$
- $r = n - 1$
- $n = 7$ and $r = 2$
- $n = 8$ and $r = 3$

This result was established using methods from algebraic topology.