Fitting the Extreme Value Theorem into a Very Compact Paper

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Abstract

This paper is the second of a pair of papers that describe the topological characteristics of the set of real numbers $\mathbb{R}$ and real-valued functions on $\mathbb{R}$, which are the underlying tools for justifying why continuity implies the existence of intermediate and extreme values. As in the first, this paper introduces some topological concepts and then follows with a concise proof of a well known calculus theorem. The Extreme Value Theorem is proven using topology.

1 Introduction

Continuity, *connectedness*, and *compactness* are recognized as three of topology's fundamental notions. This paper has three purposes: It introduces the reader to the concept of compactness; it examines the Extreme Value Theorem, a theorem commonly seen and employed in classes as early as algebra and precalculus; and it integrates these ideas by providing a topological proof of this theorem in a manner accessible by high school and college students.

In our previous paper (Cook, et al., 2016), we broadly defined *topology* as a study of spaces and distinguished between characteristics that might be considered topological as opposed to algebraic or geometric. Topological characteristics are often defined on subsets of spaces, when spaces are thought of as collections of elements. In particular, we defined *connected sets* and *open sets*. Here we restate the latter.

Definition 1.1. (Open Sets). *Let $U \subseteq \mathbb{R}$. We say that U is open if U can be obtained as the union of some collection of open intervals.*

In (Cook, et al., 2016) it was shown how the topological definition of continuity, while looking quite different on the surface, is equivalent to the classical definition of continuity. With *continuity* and *connectedness*, we proved the Intermediate Value Theorem. In this paper, we explore *compactness*. This property is required for the proof of the Extreme Value Theorem.

Theorem 1.2. (Extreme Value Theorem). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Given $a < b$, there exist $m, M \in [a, b]$ such that $f(m) \leq f(x) \leq f(M)$ for all $a \leq x \leq b$.*

Guaranteeing, that if a function defined on a closed, bounded interval is continuous, then it will achieve both a minimum and maximum value on that interval, this theorem is the theoretical bedrock on which we build all of our optimization techniques in the study of calculus. Once the definition of compactness is investigated, we will characterize the compact subsets of the real line. Then, finally, we will prove the Extreme Value Theorem.

2 Recall Continuity and Open Sets

Recall that topology includes the study of continuity and that a topologist defines continuity directly in terms of open sets instead of distances (i.e. epsilons and deltas). Again, we will focus on the topology of the real line, $\mathbb{R}$, and by *topology* we mean the collection of open sets of $\mathbb{R}$.

Equivalent to the earlier stated definition of a set that is *open*, if for each $a \in A$ we can find an open interval $I = (b, c)$ such that $a \in I$ (i.e. $b < a < c$) and $I \subseteq A$ (i.e. if $b < x < c$, then $x \in A$), then A is an open set. Intuitively this means that a set is open if given any point in that set, then other points "close by" must also belong to that set.

Open intervals are quintessential open sets. Anytime someone mentions an open set in $\mathbb{R}$, thinking of an open interval is a good first approximation. For future reference we recall the following basic properties of open sets:

- $\emptyset = \{\}$ (the emptyset) and $\mathbb{R}$ are open.
- Intersecting finitely many open sets yields an open set.
- The union of any collection of open sets yields an open set.

In fact, these three properties are what makes the collection of open sets of $\mathbb{R}$ a *topology* on $\mathbb{R}$ (known as the *standard topology*).

In our previous paper, we discussed the concept of continuity in detail. We showed that the topologist's definition of continuity matches the classical epsilon-delta definition. Here we will utilize the topological definition of continuity.

Definition 2.1. (Continuous Functions). *A function $f : \mathbb{R} \to \mathbb{R}$ is* continuous *if for every open set $V \subseteq \mathbb{R}$, we have that $f^{-1}(V) = \{x \in \mathbb{R} \mid f(x) \in V\}$ is open. Briefly, f is continuous if the pre-image of any open set, is itself open.*

3 Compactness

At first glance, compactness may seem a very strange property. It has somewhat of a checkered past, in that the current definition of compactness has not always been agreed upon as the best way of describing the property. In fact, early twentieth century mathematicians had three distinct yet equivalent definitions with which to work (Chandler, R. & Faulkner, G.). The choice of definition would often be the result of personal choice or professional training. By the mid twentieth century, these concepts were distilled down to the following rather clean definition that most topologists are brought up on today:

Definition 3.1. (Open Covers). *Let $A \subseteq \mathbb{R}$. Let $\mathcal{U}$ be a collection of open sets (note that A is a set, while $\mathcal{U}$ is a set of sets). We say that $\mathcal{U}$ is an* open cover *of A if A is contained in the union of the elements of $\mathcal{U}$ (i.e. for every $a \in A$ there is at least one $U \in \mathcal{U}$ with $a \in U$).*

As an example, the set $A = [-1, 2] \cup \{3\}$ is covered by $\mathcal{U} = \{(n, n+3) \mid n \text{ an integer}\}$. In fact, notice that just $\mathcal{U}' = \{(-2, 1), (0, 3), (1, 4)\}$ is enough to cover A. Both $\mathcal{U}$ and $\mathcal{U}'$ are open covers since they consist of open sets whose unions contains A. Notice that $\mathcal{U}'$ is a *finite* cover (it contains only 3 sets) whereas $\mathcal{U}$ is not a finite cover. Since every set belonging to $\mathcal{U}'$ belongs to $\mathcal{U}$, $\mathcal{U}'$ is called a *subcover* of A drawn from $\mathcal{U}$.

Of course, these covers are not unique. For example, $\mathcal{V} = \{(-\infty, n) \mid n = 0, 1, \dots\}$ is another open cover of A (actually $\mathcal{V}$ is an open cover of the whole real line). If we let $\mathcal{V}' = \{(-\infty, 4)\}$, we have that $\mathcal{V}'$ is a finite subcover of A drawn from $\mathcal{V}$.

Definition 3.2. (Compact Sets). *We say a set $A \subseteq \mathbb{R}$ is* compact *if given any open cover $\mathcal{U}$ there exists some natural number $n > 0$ and $U_1, \dots, U_n \in \mathcal{U}$ such that $A \subseteq (U_1 \cup \cdots \cup U_n)$. More succinctly, A is compact if* every *open cover of A contains a finite subcover of A.*

Let us reframe this definition and try to get an intuitive understanding of what compactness really is. Remember that if an open set grabs a point, then it must grab all of the points close by as well. Think of open sets grabbing points in clumps. If we cover a set with a collection of open sets, we can think of each open set grabbing clumps of points. If a set is compact, no matter how we grab out clumps of points, we should only need finitely many clumps before the set is all accounted for. In other words, with regards to the action of clumping points together, compact sets are essentially finite sets. A helpful heuristic can be: *it is often the case that if something is true about finite sets, it will also be true about compact sets.* Just as it is helpful to think of an open interval when someone mentions an open set, it is helpful to think of a finite set when one mentions a compact set.

With this heuristic in mind, the Extreme Value Theorem seems quite plausible. Recall that the Extreme Value Theorem states that continuous functions restricted to closed intervals must achieve a minimum and maximum value. If we accept that closed intervals are compact sets and then mentally replace "closed interval" with "finite set", the theorem now states that a function restricted to a finite set must achieve a minimum and maximum value. Of course, this statement is quite obvious. If we substitute a finite list of numbers into a function, we get a finite list of outputs. Some value in that list of outputs must be greatest and some value must be least.

Consider the following sets $I = [-1, 5)$ and $J = [-1, 5]$. It turns out that I is not compact, but J is. The only difference between these sets is that J contains its right endpoint, $x = 5$, while I does not. Lacking an end point ruins our chances of being compact.

To see that $I = [-1, 5)$ is not compact consider the open cover $\mathcal{U} = \{\left(-2, 5 - \frac{1}{n}\right) \mid n = 1, 2, 3, \dots\} = \{(-2, 4), (-2, 4\frac{1}{2}), (-2, 4\frac{2}{3}), (-2, 4\frac{3}{4}) \dots\}$. Since as $n \to \infty$ we have $\left(5 - \frac{1}{n}\right) \to 5$, these open intervals eventually cover up the whole set I. But if we try to cover I with any *finite* subcollection of $\mathcal{U}$ we will miss the right tail end of I. So since this open cover $\mathcal{U}$ of I has no finite subcover, I is not compact.

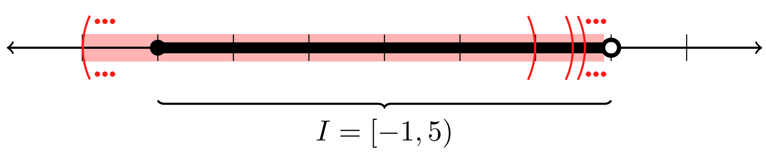

Figure 1: The half-closed interval $I = [-1, 5)$ is not compact.

Eventually we will prove that all closed, bounded intervals (i.e., closed intervals of finite length) are compact. This result is known as the Heine-Borel Theorem. For the moment, accepting that $J = [-1, 5]$ is compact, let us see how including $x = 5$ in an open cover of J changes the picture. Notice that if we tried to cover J in the same way we covered I, we would miss $x = 5$. On the other hand, as soon as we include an open set covering $x = 5$, we will grab not only $x = 5$ but also the right tail end of J. This keeps us from having the same problem that we had with I.

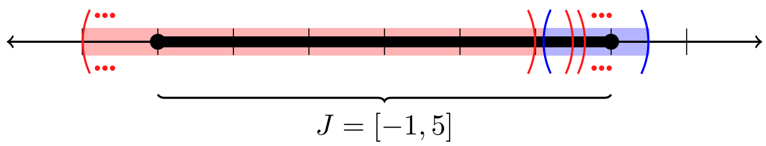

Figure 2: The closed interval $J = [-1, 5]$ is compact.

Before moving on, we should make sure we are clear about what the definition of compactness is actually saying. Compactness is *not* the same as saying we can find some finite open cover of our set or that there is some open cover that has a finite subcover. If this is what the definition said, everything would be compact. For example, notice that $\mathcal{U} = \{\mathbb{R}\}$ is an open cover of every subset of $\mathbb{R}$. Looking at $I = [-1, 5)$, we *can* cover it with finitely many open sets; $\mathcal{U} = \{(-2, 6)\}$ is an open cover of I. The definition says that a set is compact if *every* open cover has a finite subcover. So to show a set is not compact, we merely have to exhibit an open cover which has no finite subcover. Whereas, showing a set is compact means showing every open cover has a finite subcover; this is usually more difficult.

We stated earlier that compact sets can be thought to behave like finite sets. Let us prove that finite sets are indeed compact. Let $A = \{a_1, \ldots, a_n\}$ be a finite set. Next, let $\mathcal{U}$ be an open cover of A. Then since $\mathcal{U}$ covers A, given $a_i \in A$ $(1 \leq i \leq n)$ we must have some $U_i \in \mathcal{U}$ such that $a_i \in U_i$. Since each element of A belongs to some U_i $(1 \leq i \leq n)$, we have that $\{U_1, \ldots, U_n\}$ is a finite cover of A. Considering an arbitrary open cover, we have shown that it must have a finite subcover. This means A is compact.

While closed intervals of finite length make great examples of compact sets, there are other kinds of compact sets in $\mathbb{R}$ as well. For an example of an infinite set that is not an interval, consider the set $B = \{1/n \mid n = 1, 2, 3, \ldots\} \cup \{0\}$. We will show that B is compact. Let $\mathcal{U}$ be an open cover of B. Then 0 is covered by some open set in $\mathcal{U}$, say $0 \in U_0$. But since U_0 is open, it must contain some open interval $(a, b) \subseteq U_0$ with $a < 0 < b$. Notice that since b is a positive number, we can pick some natural number N such that $\frac{1}{N} < b$ (the fractions $\frac{1}{N}$ can be made arbitrarily close to 0). This implies that $\frac{1}{n} \in (a, b) \subseteq U_0$ for all $n \geq N$.

In other words, U_0 covers 0 and "most" of the sequence of fractions. We just need to cover $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{N-1}$. Since $\mathcal{U}$ covers all of B, pick some $U_j \in \mathcal{U}$ such that $\frac{1}{j} \in U_j$. Then we have that $U_1, \ldots, U_{N-1}$ cover $1, \frac{1}{2}, \ldots, \frac{1}{N-1}$. Putting this all together, $\{U_0, U_1, \ldots, U_{N-1}\}$ covers the entire set B. Again, given an arbitrary open cover $\mathcal{U}$ of B, we have found a finite subcover. This means B is compact. Intuitively, notice that if we grab a clump of points around 0, only finitely many points of B will be left, so under the action of clumping, B is more-or-less like a finite set.

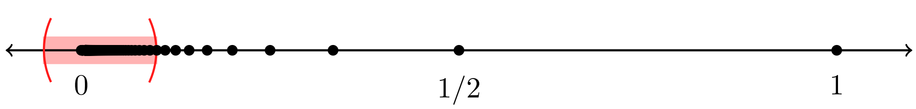

Figure 3: An open set grabbing 0 will only miss finitely many elements in B.

Let us prove a basic fact about compact sets and continuous maps. It is obvious that if we take a finite set of points and with a function map them to a set of respective images, we will still have a finite set of points. Given our heuristic that compact sets behave like finite sets, the following theorem should not be too surprising:

Theorem 3.3. (Continuous Images of Compact Sets). *Let $A \subseteq \mathbb{R}$ be compact and $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then $f(A)$ is compact. This can be more intuitively stated as, the image of a compact set under a continuous map is compact.*

Proof: Assume that A is compact. To show that $f(A) = \{f(a) \mid a \in A\}$ is compact we need to show that any open cover of $f(A)$ has a finite subcover.

Let $\mathcal{U}$ be an open cover of $f(A)$. Then $\mathcal{U}' = \{f^{-1}(U) \mid U \in \mathcal{U}\}$ is a collection of open sets in the domain of f, since $U \in \mathcal{U}$ being open implies that $f^{-1}(U)$ is also open because f is continuous.

Now let $a \in A$. Then $f(a) \in f(A)$ and so there exists some $U \in \mathcal{U}$ such that $f(a) \in U$ (because $\mathcal{U}$ covers $f(A)$). By definition $a \in f^{-1}(U)$ because $f(a) \in U$. Thus every point in A is covered by some set in $\mathcal{U}'$. Therefore, $\mathcal{U}'$ is an open cover of A.

Next, since A is compact and $\mathcal{U}'$ is an open cover of A, there must exist a finite subcover: $f^{-1}(U_1), \ldots, f^{-1}(U_n)$ (where $U_1, \ldots, U_n \in \mathcal{U}$) for A.

Notice that $A \subseteq f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_n)$ implies $f(A) \subseteq f(f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_n)) = f(f^{-1}(U_1 \cup \cdots \cup U_n)) \subseteq (U_1 \cup \cdots \cup U_n)$. Therefore, $U_1, \ldots, U_n$ is a finite subcover for $f(A)$.

Therefore, every open cover of $f(A)$ has a finite subcover, so $f(A)$ is compact. ♦

4 Heine-Borel: Characterizing Compacts Sets in $\mathbb{R}$

Before getting to the Extreme Value Theorem, we need to understand what it takes to be a compact subset of the real numbers. It turns out that there is a rather simple characterization. Namely, a subset of $\mathbb{R}$ is compact if and only if it is both *closed* and *bounded*. Needing to define both of these characteristics, we will begin by first defining and discussing boundedness.

Definition 4.1. (Bounded Sets). *Let $B \subseteq \mathbb{R}$. If for some $M \in \mathbb{R}$ we have that $M \leq b$ for all $b \in B$ (i.e., there is some real number M such that $B \subseteq [M, \infty)$), then B is* bounded below*. Likewise, if for some $K \in \mathbb{R}$ we have that $b \leq K$ for all $b \in B$ (i.e., there is some real number K such that $B \subseteq (-\infty, K]$), then B is* bounded above*. If B is both bounded below and above, we say B is* bounded.

For example, $B_1 = (-4, 20]$ is bounded. It is bounded below by $M = -4$ and above by $K = 20$. Additionally, $B_2 = [-5, 7]$, $B_3 = (-10, 17)$, and $B_4 = [-3, 2)$ are bounded intervals. On the other hand, $I = [-1, \infty)$ is bounded below but not above, so I is not a bounded interval. The set of integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is unbounded both above and below.

Lemma 4.2. (Compact Sets are Bounded). *Let $C \subseteq \mathbb{R}$ be a compact set. Then C is bounded. In other words, compact sets in $\mathbb{R}$ are bounded.*

Proof: Suppose that C is compact. Notice that $\mathcal{U} = \{(-n, n) \mid n = 1, 2, 3, \ldots\}$
$= \{(-1, 1), (-2, 2), (-3, 3), \ldots\}$ is an open cover of C. In fact, $\mathcal{U}$ covers the whole real line.

But C is compact, so $\mathcal{U}$ must have a finite subcover, say $(-n_1, n_1), \ldots, (-n_j, n_j)$. Let K be the maximum number in the list $n_1, \ldots, n_j$. Then $(-n_i, n_i) \subseteq (-K, K)$ for each $i = 1, 2, \ldots, j$. This implies that $C \subseteq (-n_1, n_1) \cup \cdots \cup (-n_j, n_j) \subseteq (-K, K)$. Thus C is bounded. ♦

The reader is invited to consider which interval drawn from $\mathcal{U} = \{(-1, 1), (-2, 2), (-3, 3), \ldots\}$ could be used as in the proof of the lemma to find bounds for previously mentioned compact sets such as $J = [-1, 5]$ (see Figure 2). The above lemma reveals that $I = [-1, \infty)$ and $\mathbb{Z}$ must not be compact sets since both are unbounded.

We note that the notion of *boundedness* is not a topological concept. Two topological spaces that are indistinguishable in the eyes of topology are said to be *homeomorphic*. While we will not precisely define the term "homeomorphic", it is not hard to show that $(0, \infty)$ and $(0, 1)$ are homeomorphic spaces. As far as topology is concerned, these two spaces are the "same". Notice that the first interval is unbounded while the second one is bounded. This means that topology cannot detect boundedness, or that the property of boundedness may not be labeled as a "topological property". Readers are invited to investigate the notion of homeomorphism as an extension to this paper (Munkres, 2000).

Moving onto the next needed characteristic, *closed*, this concept can be categorized as topological:

Definition 4.3. (Closed Sets). *Let $C \subseteq \mathbb{R}$. We say C is* closed *if $(\mathbb{R} - C) = \{x \in \mathbb{R} \mid x \notin C\}$ is open. Closed sets are precisely the complements of open sets.*

Notice that closed sets have properties complementary to the properties of open sets: $\mathbb{R} = (\mathbb{R} - \emptyset)$ and $\emptyset = (\mathbb{R} - \mathbb{R})$ are both closed, *finite* unions of closed sets are closed, and arbitrary intersections of closed sets are closed.

The sets $[-3, \infty)$, $\{1, 2, 3\} \cup [10, 20]$, and $[-12, 3]$ are all examples of closed sets (notice their complements are unions of open intervals). As a quick note, while $\mathbb{R}$ and $\emptyset$ are simultaneously both open and closed, they are the only subsets of $\mathbb{R}$ for which this is the case under the standard topology.

Lemma 4.4. (Compact Sets are Closed). *Let $C \subseteq \mathbb{R}$ be a compact set. Then C is closed. Hence, compact sets in $\mathbb{R}$ are closed.*

Proof: Suppose that C is compact. We wish to show that C is also closed or, equivalently, $(\mathbb{R} - C)$ is open.

Let $b \in (\mathbb{R} - C)$. Consider $(-\infty, x) \cup (y, \infty)$ where $x < b < y$. For each x, y with $x < b < y$, we get an *open set* surrounding b that does not contain b. If we take the union of all such sets, we will get $(-\infty, b) \cup (b, \infty) = (\mathbb{R} - \{b\})$ (the whole real line except b itself). This means that $\mathcal{U} = \{(-\infty, x) \cup (y, \infty) \mid x < b < y\}$ is an open cover of $(\mathbb{R} - \{b\})$ and because $b \notin C$, this is also an open cover of C.

Now, C is compact, therefore there exists a finite subcover: $(-\infty, x_1) \cup (y_1, \infty), \ldots, (-\infty, x_n) \cup (y_n, \infty)$ from $\mathcal{U}$ that covers C. Let $x^* = \max\{x_1, \ldots, x_n\}$ and $y^* = \min\{y_1, \ldots, y_n\}$. Then since $x_i < b < y_i$ for each $i = 1, 2, \ldots, n$, we have $x^* < b < y^*$. Also, because these intervals cover C, we have $C \subseteq \big((-\infty, x_1) \cup (y_1, \infty) \cup \cdots \cup (-\infty, x_n) \cup (y_n, \infty)\big) = \big((-\infty, x^*) \cup (y^*, \infty)\big)$. This implies that $(x^*, y^*) \subseteq (\mathbb{R} - C)$. Hence we have shown that there is an open interval contained in $(\mathbb{R} - C)$ which surrounds b. Since b was arbitrarily chosen, every element of $(\mathbb{R} - C)$ belongs to an open interval contained in this set, so that $(\mathbb{R} - C)$ is the union of open intervals. This means $(\mathbb{R} - C)$ is open. Hence C is closed. ♦

To get our characterization of compact sets, we need to recall a defining property of the real line. Recall from (Cook, et al., 2016), that the *Least Upper Bound Axiom* says that if a nonempty set $A \subseteq \mathbb{R}$ is bounded above, then A has a *least upper bound*, denoted $\text{lub}(A)$. Likewise, $\mathbb{R}$ has a logically equivalent *Greatest Lower Bound Axiom* which says if a nonempty set $A \subseteq \mathbb{R}$ is bounded below, then A has a *greatest lower bound*, denoted $\text{glb}(A)$.

For example, $A = \{\frac{1}{n} \mid n = 1, 2, 3, \dots\} = \{\dots, \frac{1}{3}, \frac{1}{2}, 1\}$ is a bounded set. In fact, $\text{lub}(A) = 1$ and $\text{glb}(A) = 0$. It is interesting to note that even when the greatest lower bound and least upper bound exist, they do not have to belong to the set. In our example, while A's least upper bound belongs to A: $1 \in A$, its greatest lower bound does not: $0 \notin A$.

We need the following lemma.

Lemma 4.5. (Closed, Bounded Sets Contain Extremes). *Let $A \subseteq \mathbb{R}$ where A is nonempty. If A is closed and bounded below, then $\text{glb}(A)$ exists and $\text{glb}(A) \in A$. Likewise, if A is closed and bounded above, then $\text{lub}(A)$ exists and $\text{lub}(A) \in A$. Briefly, for any closed set, if a greatest lower bound or least upper bound exists, it must belong to that set.*

Proof: Suppose that A is closed and bounded below. Then by the Greatest Lower Bound Axiom, $b = \text{glb}(A)$ exists. For sake of contradiction, suppose that $b \notin A$.

We have assumed that A is closed, so by definition its complement $A^c = (\mathbb{R} - A)$ is open. Since we have assumed $b \notin A$, b belongs to the open set A^c. This implies there is an open interval containing b which is itself contained in A^c. Specifically, there exists some $(a, c) \subseteq A^c$ where $a < b < c$.

Consider $x \in A$, then $b \leq x$ because b is a lower bound for A. It is impossible to have $x < c$ because this would imply that $a < b \leq x < c$ which means $x \in (a, c)$ and so $x \in A^c$ (which contradicts the fact that $x \in A$). Therefore, we must have $c \leq x$.

We have just shown that $c \leq x$ for any and all $x \in A$. Therefore, c is a lower bound for A. But also $b < c$. Therefore, b is *not* the *greatest* lower bound. This contradicts the definition of b. Therefore, we are forced to conclude that $b \in A$.

A very similar proof establishes the analogous result for least upper bounds. ♦

Now we are ready to state our theorem about the compact subsets of $\mathbb{R}$. This result is known as the Heine-Borel Theorem.

Theorem 4.6. (Heine-Borel). *Let $A \subseteq \mathbb{R}$. A is compact if and only if A is closed and bounded.*

Before heading into the proof of the Heine-Borel theorem, let us consider a few examples. The interval $A_1 = (-1, 2]$ is bounded, but it is not closed and so by the Heine-Borel theorem A_1 is not compact. The interval $A_2 = (-\infty, 5]$ is closed because its complement $(\mathbb{R} - A_2) = (5, \infty)$ is open, but A_2 is not bounded, so A_2 is not compact. On the other hand, sets like $A_3 = [-2, 9]$ and $A_4 = [-1, 0] \cup \{1, 2, 3\} \cup [8, 99]$ are both closed and bounded, so both A_3 and A_4 are compact.

As a note to the reader, the following proof is more technical than the other proofs in this paper. We include this proof for completeness, but it can be safely skipped without damaging the reader's understanding of compactness. We recommend skimming this proof during a first reading.

Proof: We have already proven the easier direction of this theorem. Lemmas 4.4 and 4.2 tell us that if A is compact, then it must be both closed and bounded. The other direction of this proof gets a bit more technical.

Note, if A is empty, then A is compact ($A = \emptyset$ is certainly finite and thus compact). We will now assume that A is non-empty.

Suppose A is closed and bounded. Let $\mathcal{U}$ be an open cover of A. Since A is bounded, the Greatest Lower Bound and Least Upper Bound Axioms guarantee that both $m = \text{glb}(A)$ and $M = \text{lub}(A)$ exist. Also, A is closed, so by Lemma 4.5 $m, M \in A$.

Our goal is to show that $\mathcal{U}$ has a finite subcover for A. To this end, consider the set $B = \{b \in \mathbb{R} \mid m \leq b \text{ and } ([m,b] \cap A) \text{ is finitely covered by } \mathcal{U}\}$. Before moving on, let us consider exactly what the set B is. Given any $b \geq m$, $([m,b] \cap A)$ is the part of the set A which lies at or below b on the real number line. Moreover, if $b \in B$, there exists finitely many sets in $\mathcal{U}$ whose union contains $([m,b] \cap A)$. More-or-less, B is keeping track of how much of A can be finitely covered by $\mathcal{U}$.

Clearly, $m \in B$ since $([m,m] \cap A) = \{m\}$ is finitely covered by any single set in $\mathcal{U}$ that contains m. Thus B is non-empty.

If this theorem is true, *all* of A can be finitely covered by $\mathcal{U}$, so all $b \geq M$ belong to B (for such b's we have $([m,b] \cap A) = A$). In other words, B *should* be a set that is not bounded above. So for sake of contradiction, we assume that $t = \text{lub}(B)$ exists (i.e., that B is bounded above).

Case 1: This least upper bound of B is either in A or it is not, so first suppose $t \in A$. Then there exists some $V \in \mathcal{U}$ such that $t \in V$. But V is open so there exists some open interval $(p,q) \subseteq V$ such that $p < t < q$.

Now t is the least upper bound for B. If there is no element in B between p and t, everything between p and t would be an upper bound for B contradicting t being the *least* upper bound. Therefore, we can choose some $s \in B$ such that $p < s < t$. But $s \in B$ implies that $([m,s] \cap A)$ can be finitely covered by some $U_1, \ldots, U_N \in \mathcal{U}$. Adding V to this collection (i.e., $U_1, \ldots, U_N, V$) produces a finite subcover of $([m,r] \cap A)$ for any r with $t < r < q$. Therefore, $r \in B$ for all $t < r < q$ which contradicts t being an upper bound. Thus t cannot belong to A.

Case 2: Suppose $t \notin A$. Then since A is closed, $(\mathbb{R} - A)$ is open and so we can find an open interval $(p,q) \subseteq (\mathbb{R} - A)$ such that $p < t < q$. Recall that t is the least upper bound for B. If there is no element in B between p and t, everything between p and t would be an upper bound for B contradicting t being the *least* upper bound. Therefore, we can choose some $s \in B$ such that $p < s < t$.

But again, $s \in B$ implies $([m,s] \cap A)$ can be finitely covered by some $U_1, \ldots, U_N \in \mathcal{U}$. Further, $(p,q) \subseteq (\mathbb{R} - A)$, so $([m,s] \cap A) = ([m,r] \cap A)$ for any $p < r < q$. Notice A doesn't contain any points in (p,q) so changing r from s within this interval has no effect on the intersection. Thus we have that $U_1, \ldots, U_N$ also covers $([m,r] \cap A)$ for all $s < r < q$. In particular, each r where $t < r < q$ belongs to B since $([m,r] \cap A)$ can be finitely covered by $U_1, \ldots, U_N$. Thus t is not an upper bound for B (contradiction).

Both $t \in A$ and $t \notin A$ led to contradictions. This means t cannot exist. In other words, B cannot have a least upper bound. This means B is not bounded above.

Finally, consider $M = \text{lub}(A)$. Since B is not bounded above, there is some $s \in B$ such that $s > M$. Thus $A = ([m,M] \cap A) = ([m,s] \cap A)$ is finitely covered by $\mathcal{U}$. Therefore, A is compact. ♦

Corollary 4.7. (Closed, Bounded Intervals are Compact). *Consider $a, b \in \mathbb{R}$ such that $a \leq b$. Closed, bounded intervals $I = [a,b]$ are compact.*

Proof: Notice that $(\mathbb{R} - I) = (-\infty, a) \cup (b, \infty)$ so $(\mathbb{R} - I)$ is a union of open intervals making it an open set. Therefore, I is closed. It makes sense that a *closed* interval should be a closed set. Also, I is bounded by a and b. Therefore, by Theorem 4.6 (the Heine-Borel theorem), I is compact. ♦

We can combine the Heine-Borel theorem characterizing compact subsets of $\mathbb{R}$ with our theorem in the previous paper (Cook, et al., 2016) characterizing connected subsets of $\mathbb{R}$. Theorem

4.6 (the Heine-Borel theorem) states that I is compact if and only if I is closed and bounded. Theorem 4 (Intervals are Connected) in (Cook, et al., 2016) states that I is connected if and only if I is an interval. Thus I is compact and connected if and only if I is closed, bounded, and an interval.

Corollary 4.8. (Compact, Connected Subsets of $\mathbb{R}$). *Let $I \subset \mathbb{R}$. The set I is compact and connected if and only if $I = [a, b]$ for some $a, b \in \mathbb{R}$ with $a \leq b$. In words, the only compact and connected subsets of the real numbers are the closed intervals of finite length.*

$$
\begin{array}{ccc}
\text{compact} & \iff & \text{closed and bounded} \\
+ & & + \\
\text{connected} & \iff & \text{interval} \\
\| & & \| \\
\text{compact and connected} & \iff & \text{closed, finite length interval}
\end{array}
$$

5 The Extreme Value Theorem

We can now synthesize our results into the culminating idea, the original purpose of this paper.

Theorem 5.1. (Extreme Value Theorem). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Given $a \leq b$, there exist $m, M \in [a, b]$ such that $f(m) \leq f(x) \leq f(M)$ for all $a \leq x \leq b$.*

Proof: The closed, bounded interval $I = [a, b]$ is compact by Corollary 4.7. Theorem 3.3 states that compact sets map to compact sets under continuous functions. Therefore, we have $f(I)$ is compact. Theorem 4.6 then tells us that $f(I)$ is closed and bounded. In particular, by Lemma 4.5, $\text{glb}(f(I)) \in f(I)$ and $\text{lub}(f(I)) \in f(I)$.

This means that $\text{glb}(f(I)) = f(m)$ and $\text{lub}(f(I)) = f(M)$ for some $m, M \in I$. Which then by the definition of greatest lower bound and least upper bound implies that $f(m) \leq f(x) \leq f(M)$ for all $x \in I$. ♦

In our statement of the theorem, we have introduced the assumption that f is continuous on the whole real line. This allows us to avoid defining continuity on subspaces of $\mathbb{R}$ which would make this paper unnecessarily cumbersome. This "weakening" of the theorem can easily be remedied by taking f defined on a closed interval $[a, b]$ and extending it to a function defined on all $\mathbb{R}$ as follows: $f(x) = f(a)$ for $x < a$, $f(x) = f(x)$ for $a \leq x \leq b$, and $f(x) = f(b)$ for $x > b$. This extends a continuous function defined on $[a, b]$ to a continuous function defined on all $\mathbb{R}$ (then all of our results can be applied).

We conclude with a final *improved* version of the Extreme Value Theorem that invokes results from our previous paper (Cook, et al., 2016).

Theorem 5.2. (Extreme Value Theorem 2). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Given $a \leq b$, there exist $c \leq d$ such that $f([a, b]) = [c, d]$.*

Proof: By Corollary 4.8 $[a, b]$ is both compact and connected. Theorem 3.3 along with Theorem 3 (Continuity Preserves Connectedness) in (Cook, et al., 2016) then tells us that $f([a, b])$ is both compact and connected. Applying Corollary 4.8 then gives us that $f([a, b]) = [c, d]$ for some $c, d \in \mathbb{R}$ with $c \leq d$. ♦

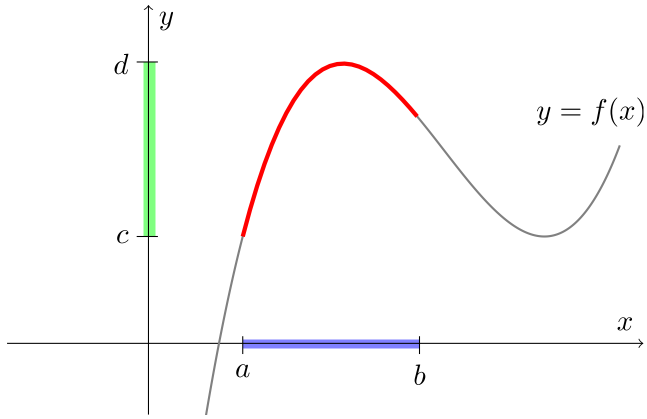

Figure 4: Continuous functions send closed intervals to closed intervals: $f([a, b]) = [c, d]$.

In summary, this current paper along with its predecessor (Cook, et al., 2016) have provided topological examinations of two theorems (i.e., the Intermediate and Extreme Value Theorems) commonly seen as early as precalculus. While many students find such theorems to be intuitively obvious, we now hope the reader better appreciates the role topology plays in the establishment of these theorems. For further exploration of the fascinating subject of topology, may we recommend James Munkres' excellent text (Munkres, 2000).

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