Simplicity and Beauty of Topology: Connecting with the Intermediate Value Theorem

William J. Cook, Katherine J. Mawhinney, & Michael J. Bossé

October 12, 2018

Abstract

Topological concepts are foundational and provide structural support for ideas commonly investigated in high school mathematics, particularly continuity and the Intermediate Value Theorem. Herein, we endeavor to introduce topological concepts to calculus teachers or to highly motivated students of calculus, and connect these concepts to the standard definition of continuity. We then introduce connectedness and prove the Intermediate Value Theorem. It is hoped that a topologically motivated proof will provide greater insight than the more commonplace epsilon-delta proofs associated with the Intermediate Value Theorem.

1 Introduction

Although high school students have typically never heard of the mathematical field of topology and very few undergraduate mathematics majors have experienced more than its most brief introduction, these students regularly investigate concepts born from topology. In fact, so broad and far reaching is the field of topology that its fundamental notions inherently reside in, and intersect with, almost all other mathematical fields. Unfortunately, due to its abstract nature, topology as a whole has historically been deemed most appropriate for graduate mathematics students. However, seminal notions within topology are readily within the grasp of instructors who teach calculus and motivated calculus students, and these topics aid in explaining introductory calculus concepts.

Despite topology being ubiquitous through its mathematical connections, it can be difficult to succinctly define. From its root words, we could say that topology is the study of spaces. However, absent from this overly terse definition is that within this field of study lies the development of unique characteristics of spaces that can only be labeled as *topological*, as opposed to geometric or algebraic. This ideational extension provides a powerful definition that, rather than moving us in the direction of any one of the many branches within mathematics, moves us to be able to intersect with all of mathematics.

Now we begin to examine a few topics in high school mathematics and first semester calculus from a topological perspective. We begin by investigating the definition of continuity and then apply this to the Intermediate Value Theorem, implicitly used in precalculus and formally introduced in a first differential calculus course preceding derivatives. Notably, embedded within more common epsilon-delta proofs of the Intermediate Value Theorem are the topological constructs that will be presented herein.

Theorem 1.1 (Intermediate Value Theorem). *Suppose f is continuous on a closed interval $[a, b]$. If y is any number between $f(a)$ and $f(b)$, then there is at least one number x in $[a, b]$ such that $f(x) = y$.*

Immediately following the presentation of this theorem, most textbooks provide a comment such as:

> *An important property of continuous functions is expressed by the following theorem, whose proof is found in more advanced books on calculus.* (Stewart, 2015)

A textbook might also hint at why the proof is not included within, such as

> *Although this theorem is intuitively obvious, its proof depends on a mathematically precise development of the real number system, which is beyond the scope of this text.* (Anton, et al., 2012)

Taalmon and Kohn proffer the following of both the Intermediate and Extreme Value Theorems:

> *These two important consequences of continuity may seem obvious, but in fact they rely on a subtle mathematical property of the real numbers called the* **Least Upper Bound Axiom**. *Properly explaining the proofs of these theorems is outside the scope of this book.* (Taalman, et al., 2013)

These comments provide hints pointing to the characteristics of the domain of the function, f, in the Intermediate Value Theorem; in calculus, the domain of a function is commonly assumed to be a subset of the real numbers, $\mathbb{R}$. Such a set inherits structure from the *topological space*' of real numbers, so the domain of f is more than a mere set; it can be viewed as a topological space. Intervals of real numbers possess a powerful topological property known as *connectedness*. Continuity is a fundamental notion of topology. In the Intermediate Value Theorem, the requirement of the domain to be an interval, the continuity of f, and the range lying inside the set of real numbers all link this theorem to the topic of topology.

The preceding paragraph is rife with nomenclature that may be new to many calculus students. Therefore, in the following discussions we will define and apply terms such as continuity, open and closed intervals, least upper bound, and connectedness. To accomplish this, we begin with a discussion of continuity moving from a standard epsilon-delta presentation to a topological perspective.

2 Continuity

In its generality, topology is quite powerful; but generality can unnecessarily obscure the simplicity of the concepts involved. In this paper, discussion is constrained to a single topological space: the real numbers, $\mathbb{R}$.

The following common definition of a continuous function appears in most calculus textbooks.

Definition 2.1 (Continuity). *A function $f : \mathbb{R} \to \mathbb{R}$ is* continuous *if for every $a \in \mathbb{R}$ and for every $\epsilon > 0$ there exists some $\delta > 0$ such that $|x - a| < \delta$ implies that $|f(x) - f(a)| < \epsilon$. See Figure 1 for an illustration of this definition.*

This rigorous definition, first penned by Karl Weirstrass in 1872, was long overdue. Almost fifty years prior, Augustin-Louis Cauchy's 1821 treaty *Cours d' analyze de l' Ecole Royale Polytechnique* put in words the closeness expressed in Weirstrass's definition with phrases such as "indefinitely approach to a fixed value... differing from it by as little we wish" (Bradley & Sandifer, 2009). Cauchy's work is heralded as the first treatment of calculus with analytic proofs that did not rely on geometric notions. Before Cauchy, mathematicians relied on ideas similar to the ubiquitous *can be drawn without picking up your pencil* notion of continuity that works for most examples but is not always true.

While we could focus on continuity at a single point (by picking some fixed real number $a \in \mathbb{R}$), for simplicity we will simply assume that our function is continuous everywhere. This avoids discussions about subspace topologies and neighborhood bases that would unnecessarily clutter and complicate this current investigation.

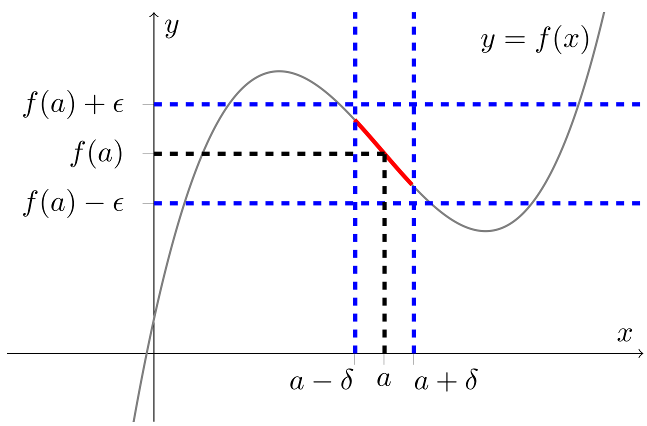

Figure 1. Epsilon-Delta Continuity at $x = a$.

Now let us focus on the inequalities at the end of this definition of continuity. We note that $|x - a| < \delta$ is the same as $a - \delta < x < a + \delta$, which again is the same as $x \in (a - \delta, a + \delta)$. Likewise, $|f(x) - f(a)| < \epsilon$ can be replaced with $f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$. Essentially, our definition says that given an open interval (p, q) containing $f(a)$, we can find an open interval (c, d) such that $x \in (c, d)$ implies $f(x) \in (p, q)$. So, continuity can be recognized as simply a statement regarding open intervals. While open intervals are a special kind of set and could be used to accomplish our task, it is more convenient to work with the notion of an *open set*, a concept that will be foundational to our following discussions.

Definition 2.2 (**Open Sets**). *Let $U \subseteq \mathbb{R}$. We say that U is open if U can be obtained as the union of some collection of open intervals. Briefly, U is a union of open intervals.*

The preceding definition implies that open intervals themselves are open sets, so $(-1, 3)$ is an open set. Notice $(-5, \infty) = (-5, 1) \cup (-5, 2) \cup (-5, 3) \cup \cdots$; so *unbounded* (open) intervals are open sets. Likewise, $\mathbb{R}$ itself is open. Since the interval (a, a) equals the empty set, $\emptyset = \{\}$, we have that the empty set is an open set (note that $(a, a) \neq \{a\}$). The following summarizes some basic properties of open sets:

- $\emptyset$ and $\mathbb{R}$ are open.
- Intersecting finitely many open sets yields an open set.

- The union of (any) collection of open sets yields an open set.

We must be careful not to allow infinite intersections, however. Consider the following example: $(-1,1)\cap(-1/2,1/2)\cap(-1/3,1/3)\cap\cdots = \{0\}$. Now $\{0\}$ is not open, since it cannot be realized as a union of open intervals. Thus, infinite intersections of open sets are not necessarily open.

It is worth commenting at this point that any collection of subsets of X (some fixed set) with the above properties (where $\mathbb{R}$ is replaced by X) is called a *topology* for X. The set X equipped with some *topology* is called a *topological space*.

Now it is time to revisit our definition of continuity. Recall that $f : \mathbb{R} \to \mathbb{R}$ is continuous if for each $a \in \mathbb{R}$ and any open interval (p, q) with $f(a) \in (p, q)$, there exists some open interval (c, d) such that $x \in (c, d)$ implies that $f(x) \in (p, q)$. This means that any element mapping into (p, q) must belong to an open interval of elements mapping into (p, q). Therefore, the collection of elements which map into (p, q) must be a union of open intervals–that is, an open set! We can generalize a little bit and notice that any union of intervals like (p, q) must be mapped to by a union of open sets. Notice that we have supplanted the more commonplace epsilon-delta definition of continuity with the consideration of open intervals. We have now motivated the following definition:

Definition 2.3 (**Topological Continuity**). *A function $f : \mathbb{R} \to \mathbb{R}$ is* continuous *if for every open set $V \subseteq \mathbb{R}$, we have that $f^{-1}(V) = \{x \in \mathbb{R} \mid f(x) \in V\}$ is open. Briefly, f is continuous if the inverse image of any open set, is itself open (see Figure 2).*

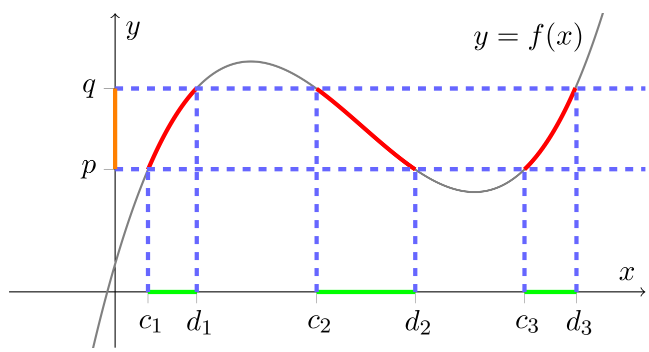

Figure 2. Topological Continuity: $f^{-1}(p,q) = (c_1,d_1) \cup (c_2,d_2) \cup (c_3,d_3)$.

From our discussion, hopefully it is clear that our new definition is really just the old one in disguise (i.e., they are equivalent). To demonstrate the difference between a classical analytical proof and a topological proof, consider the following theorem.

Theorem 2.4 (**Continuous Composition**). *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Then $g \circ f : \mathbb{R} \to \mathbb{R}$ is continuous. Briefly, the composition of two continuous functions, is itself continuous.*

This motivates our first topological proof. However, in order to recognize the aesthetic value of a topological proof, we compare it with a more frequently seen analytic proof of continuous composition. We offer the analytic proof first.

Analytic Proof. Let $a \in \mathbb{R}$. Suppose $\epsilon > 0$. Then g is continuous at $y = f(a)$, so there exists some $\gamma > 0$ such that $|y - f(a)| < \gamma$ implies that $|g(y) - g(f(a))| < \epsilon$. Next, since f is continuous at $x = a$, there exists some $\delta > 0$ such that $|x - a| < \delta$ implies that $|f(x) - f(a)| < \gamma$. Therefore, $|x - a| < \delta$ implies that $|f(x) - f(a)| < \gamma$ which in turn implies that $|g(f(x)) - g(f(a))| < \epsilon$. This means $g \circ f$ is continuous at $x = a$ for all $a \in \mathbb{R}$. ♦

Topological Proof. Let $V \subseteq \mathbb{R}$ be an open set. Then $U = g^{-1}(V)$ is open since g is continuous. We have that $f^{-1}(U)$ is open since f is continuous. Therefore, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) = f^{-1}(U)$ is open for every open set V. This means that $g \circ f$ is continuous. ♦

So, what have we accomplished? First, we can clearly see that the topological proof is more succinct than the analytical proof. This is, in and of itself, quite elegant. However, we can also begin to notice that topological notions are secreted in the analytical proof. While the analytical proof employs epsilons and deltas, we have recognized that this is synonymous with intervals. Thus, not only can we start to see the potential for topology to simplify proofs and clarify ideas, we see that topology is inherently embedded in some of these ideas. To investigate this further, let us consider the idea of *connectedness*.

3 Connectedness

The Intermediate Value Theorem not only involves the concept of continuity as previously discussed but also connectedness. If asked what is meant by *connected*, one might intuitively answer *it has one piece*. While this may informally capture some of the idea, topology provides us a mechanism through which to clarify this notion.

Definition 3.1 (Separated and Connected Sets). *Let $A \subseteq \mathbb{R}$. A pair of open sets $U, V \subseteq \mathbb{R}$ is called a* separation *of A if*

(a) $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$ (i.e., U and V both contain some points in A);

(b) $A \subseteq U \cup V$ (i.e., U and V together cover *all of A); and*

(c) $U \cap V = \emptyset$ (i.e., U and V are disjoint).

If A has a separation, it can be separated*. If A can be separated, it is* not connected*. If A cannot be separated (i.e., no separation exists), then A is* connected *(see Figure 3).*

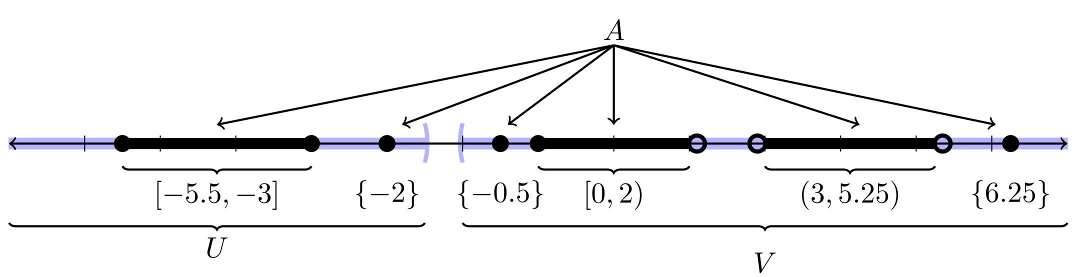

Figure 3. The set A is separated by open sets U and V.

In Figure 3, $A = [-5.5, -3] \cup \{-2\} \cup \{-0.5\} \cup [0, 2) \cup (3, 5.25) \cup \{6.25\}$ is a subset of the set of real numbers, $\mathbb{R}$. Note that A can be separated by the open sets $U = (-\infty, -1.5)$ and $V = (-1, \infty)$, so A is not a connected set. On the other hand, the sets $[-5.5, -3]$, $\{-2\}$, $\{-0.5\}$,

$[0,2)$, $(3,5.25)$, and $\{6.25\}$ are all examples of connected sets. It turns out that all subsets can be disassembled into such *connected components*. If one takes the time to formalize this, we get that a set is connected if and only if it has a single connected component.

Consider another example, the integers $\mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\}$ form a disconnected subset of the real numbers. In particular, we can separate $\mathbb{Z}$ using open sets such as $(-\infty,1/2)$ and $(1/2,\infty)$. Similarly the rational numbers $\mathbb{Q} = \{a/b \mid a,b \in \mathbb{Z} \text{ and } b \neq 0\}$ are another disconnected subset of the reals. Here, we can separate with open sets such as $(-\infty,\sqrt{2})$ and $(\sqrt{2},\infty)$.

Intervals are connected. For example, $[-1,2]$, $(3,\infty)$, the empty set, and $\mathbb{R}$ itself are connected. Notice that we can separate an interval such as $[-1,2]$ with intervals: $[-1,2] \subseteq (-5,1) \cup [1,5)$; but we cannot separate $[-1,2]$ with *open* intervals. While this might be intuitively obvious (mathematicians for hundreds of years thought so), this fact requires proof (which we will supply later).

Parenthetically, although we have already covered numerous topological concepts and proofs, we have endeavored to accomplish this in a manner accessible to students of calculus who read attentively and carefully. This again argues that topology, at least at an introductory level, can be investigated by a far wider audience than those who most commonly encounter topological ideas.

In the preceding paragraphs, we have defined continuity, proven continuous composition, and investigated connectedness. With an eye toward developing the necessary fundamental aspects of the Intermediate Value Theorem, we will now prove an interesting theorem stating that continuity preserves connectedness. Without this theorem, we could not guarantee that every intermediate function value could be achieved from the continuous function's domain.

Theorem 3.2 (Continuity Preserves Connectedness). *Let $A \subseteq \mathbb{R}$ be a connected set and $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. Then $f(A) = \{f(x) \mid x \in A\}$ is connected.*

Proof: Let us assume that $f(A)$ is not connected (and derive a contradiction). Suppose that U, V is a separation of $f(A)$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are open sets (since f is continuous). We now show that $f^{-1}(U), f^{-1}(V)$ separates A (see Definition 3.1).

3.1(a) Since $U \cap f(A) \neq \emptyset$ there is some $a \in A$ such that $f(a) \in U$. Thus $a \in A \cap f^{-1}(U)$ so $A \cap f^{-1}(U) \neq \emptyset$. Likewise, $A \cap f^{-1}(V) \neq \emptyset$.

3.1(b) $f(A) \subseteq U \cup V$. Thus $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$.

3.1(c) $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$.

Therefore, $f^{-1}(U), f^{-1}(V)$ separates A (contradicting our assumption that A is connected). Thus, no separation of $f(A)$ can exist; so we have that $f(A)$ is connected. ♦

With these fundamental components in place, there yet remains one more task before we attack the Intermediate Value Theorem. We need to confirm that intervals are connected and that every connected subset of $\mathbb{R}$ is an interval. Before embarking on the proof of this statement, we must introduce the concept of a *least upper bound* (or supremum).

Given a set of real numbers $A \subset \mathbb{R}$, we say that $r \in \mathbb{R}$ is an *upper bound* of A if $x \leq r$ for all $x \in A$. Note that r does not have to belong to the set A. For example, the number $r = 5.5$ is clearly an upper bound for open interval $A = (-3,2)$. We say that r is a *least upper bound* for a set A if r is an upper bound and given any other upper bound s, we have $r \leq s$. Quite literally, r is the least of all of the upper bounds (see Figure 4).

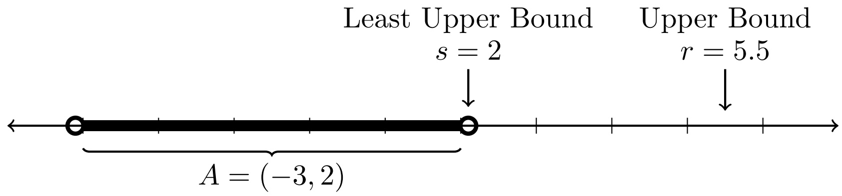

Figure 4. The set $A = (-3, 2)$ is bounded above by $r = 5.5$ with least upper bound $s = 2$.

One of the most important properties of the real numbers is its **Least Upper Bound Axiom**. This property of $\mathbb{R}$ gives it much of its topological attributes. The axiom states that given any *non-empty* subset $A \subseteq \mathbb{R}$, if A has an upper bound, then A must have a least upper bound. In the case that a least upper bound exists, we denote it by $\text{lub}(A)$.

For example, $A = (-3, 2)$ is bounded above, so its least upper bound must exist. Clearly, $\text{lub}(A) = 2$. Notice that while A is guaranteed to have a least upper bound, it is not required that $\text{lub}(A)$ is actually a member of A, as is the case with $A = (-3, 2)$. Also, let b_n be π truncated after the n^{th} digit. Consider $B = \{b_n \mid n = 0, 1, 2, \ldots\}$. This set is bounded above by 4, so the axiom tell us that the least upper bound for B exists. In fact, $\text{lub}(B) = \pi$. Furthermore, sets like $(0, \infty)$, $\mathbb{Z}$, and $\mathbb{R}$ are not bounded above, so their least upper bounds do not exist. We could also consider lower bounds and greatest lower bounds (infimums), but upper bounds and least upper bounds suffice for what follows.

Theorem 3.3 (Intervals are Connected). *Let $I \subseteq \mathbb{R}$. I is connected if and only if I is an interval.*

As mentioned above, the empty set is an interval. Also, we allow unbounded intervals and make no assumptions about end points (so $[-2, 5]$, $[-1, 0)$, $(-\infty, 3]$, and $\{5\} = [5, 5]$ are all included under the term *interval*).

Proof: Since the theorem stating that intervals are connected is a biconditional (e.g., if and only if), it must be proven in both directions. First, we will prove that if I is connected, then it must be an interval. This direction is quite easy if we do so using the contrapositive.

Suppose that I is not an interval. Then there exists $a < b < c$ such that $a, c \in I$ but $b \notin I$. Consider $U = (-\infty, b)$, $V = (b, \infty)$. Clearly, $a \in U$ and $c \in V$ (so $U \cap I \neq \emptyset$ and $V \cap I \neq \emptyset$ meeting Definition 3.1(a)). Equally clear: $I \subseteq U \cup V = (-\infty, b) \cup (b, \infty)$ (meeting Definition 3.1(b)), since $b \notin I$. Finally, $U \cap V = \emptyset$ as well (meeting Definition 3.1(c)). Thus U, V is a separation of I, so I is not connected.

Briefly, this half of the proof states that in the case that I is not an interval, we can break I into two pieces using any point I "skips over" as a dividing point. It may be helpful to refer back to Figure 3.

For the other direction of this proof, suppose that I is an interval. We must work a little harder in this portion of the proof since it requires the Least Upper Bound axiom along with the completion of a number of smaller tasks.

Suppose that I is separated by U, V; we will now derive a contradiction. Since $U \cap I \neq \emptyset$ and $V \cap I \neq \emptyset$ (Definition 3.1(a)), there exists some $a \in U \cap I$ and $c \in V \cap I$. Without loss of generality, we can assume $a < c$. (Note: We cannot have $a = c$ since U and V are disjoint.)

Consider the set $\mathcal{B} = \{y \in \mathbb{R} \mid (a, y) \subseteq U\}$. The idea here is to see how far we can depart from a without leaving the open set U. Notice that since $a \in U$ and U is open (i.e., a union of open intervals) there is some interval $(p, q) \subseteq U$ with $a \in (p, q)$. Thus, $(a, q) \subset (p, q) \subseteq U$, so $q \in \mathcal{B}$. Therefore, $\mathcal{B}$ is non-empty (departing a least slightly from a does not leave U). Next, $\mathcal{B}$ is bounded above by c since, otherwise we would have some $d \in \mathcal{B}$ such that $d > c$ and so $c \in (a, d) \subseteq U$. In particular, $c \in U$ so that U and V are not disjoint (contradicting Definition 3.1(c)). Briefly, $\mathcal{B}$ must be bounded above since $c \in V$; trying to extend (a, c) would cause us to leap outside of U. In summary, $\mathcal{B}$ is a non-empty set of real numbers bounded above by c.

We now call on the Least Upper Bound axiom: If $\mathcal{B}$ is non-empty and bounded above (which it is), then $\text{lub}(\mathcal{B}) = b$ exists. Notice that if r lies between a and b, then $(a, r) \subseteq U$. (If $(a, r) \not\subseteq U$, then r would yield a smaller upper bound for $\mathcal{B}$. However, b is the *least* upper bound for $\mathcal{B}$.) Therefore, all open intervals (a, r) for $a < r < b$ are contained in U. Since U is an open set, the union of such open intervals must also be contained in U. The union of all (a, r) for $a < r < b$ is nothing more than the open interval (a, b). Therefore, $(a, b) \subseteq U$. This means that $b \in \mathcal{B}$ ($\mathcal{B}$ contains its least upper bound).

Now b is the least of all upper bounds for $\mathcal{B}$ and c is an upper bound, so $b \leq c$. Next, since b is an upper bound for $\mathcal{B}$, $a \leq b$. This leads to the question: Where does b belong? Notice that b lies between a and c, so $b \in I$ (since $a, c \in I$ and I is an interval–we finally used that I is an interval!). Recall that $I \subseteq U \cup V$ (Definition 3.1(b)). Therefore, either $b \in U$ or $b \in V$. We have almost reached the end. In the next paragraph, we will find that both $b \in U$ and $b \in V$ lead to a contradiction.

First, suppose $b \in U$. Then since U is an open set (a union of open intervals) there exists some open interval (p, q) containing b which lies inside of U. However, $(a, b) \subseteq U$ (because $b \in \mathcal{B}$) and so $(a, q) \subseteq (a, b) \cup (p, q) \subseteq U$. In other words, (a, b) can be extended a bit further and still stay inside U. This means that $q \in \mathcal{B}$ and $q > b$. Thus, b is not an upper bound for $\mathcal{B}$. This contradicts the definition of b which states $b = \text{lub}(\mathcal{B})$. Thus b cannot possibly belong to U.

Therefore, we must have that $b \in V$. However, like U, V is open so there is some open interval (p, q) containing b which lies inside of V. This means that $p < b < q$, so the intervals (a, b) (contained in U) and (p, q) (contained in V) intersect. This means that U and V overlap: $U \cap V \neq \emptyset$ (contradicting Definition 3.1(c)). Thus, b cannot belong to V.

With all cases covered, no separation of I can exist (i.e., I is connected). ♦

We have finally developed all the necessary topological components to investigate the Intermediate Value Theorem–the goal of this paper. With these pieces in hand, the proof of this theorem is quite simple, and indeed elegant–arguably far more elegant than more common epsilon-delta proofs that conceal their topological roots.

Theorem 3.4 (Intermediate Value Theorem). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $a < b$. Then given any real number y between $f(a)$ and $f(b)$, there exists some $a \leq x \leq b$ such that $f(x) = y$.*

Proof: $I = [a, b]$ is an interval, so it is connected (Theorem 3.3). Therefore, $f(I)$ is connected (since f is continuous; see Theorem 3.2). This means that $f(I)$ is an interval (Theorem 3.3). Since $f(a), f(b) \in f(I)$ and $f(I)$ is an interval, if y lies between $f(a)$ and $f(b)$, then $y \in f(I)$. Therefore, there exists some $x \in I = [a, b]$ such that $f(x) = y$. ♦

The careful reader will recognize that we have altered the hypothesis of the original Intermediate Value Theorem (Theorem 1.1). We have introduced the assumption that f is continuous on the whole real line. This allows us to avoid defining continuity on subspaces of $\mathbb{R}$ which would make this paper unnecessarily cumbersome.

This "weakening" of the theorem can easily be remedied by taking f defined on a closed interval $[a, b]$ and extending it to a function defined on all $\mathbb{R}$ as follows: $f(x) = f(a)$ for $x < a$, $f(x) = f(x)$ for $a \leq x \leq b$, and $f(x) = f(b)$ for $x > b$. This extends a continuous function defined on $[a, b]$ to a continuous function defined on all $\mathbb{R}$ (then Theorem 3.4 applies).

We have achieved our goal: We proved the Intermediate Value Theorem using the sophisticated beauty of topology. We accomplished this in a manner accessible to instructors of introductory calculus and motivated calculus students who may or may not have had previous instruction in topology. Thus, in addition to proving this exquisite theorem, we have demonstrated that some topological concepts are well within reach of a wide audience of calculus teachers and students. It is hoped that this will develop some intrigue in readers to investigate topology to a greater extent. May we recommend James Munkres' excellent text (Munkres, 2000).

References

Anton, H., I. Bivens, *et* al. (2012). *Calculus Single, 10th edition.* Wiley.

Bradley, R.E., & Sandifer, C.E. (2009). *Cauchy's cours d'analyse: An annotated translation.* In J.Z. Buchwald (Series Ed.) Sources and studies in the history of mathematics and physical sciences. New York, NY: Springer.

Cauchy, A. (1821). *Cours d'analyse de l'École Royale Polytechnique.* Paris, France: Imprimerie Royale.

Munkres, J. (2000). *Topology, 2nd edition.* Prentice Hall.

Stewart, J. (2015). *Calculus, 8th edition.* Brooks Cole.

Taalman, L. and P. Kohn. (2013). *Calculus* W.H. Freeman and Company.