Finding Real Roots of Polynomials Using Sturm Sequences

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Abstract

Following a line of inquiry regarding the exact number of real roots of a real polynomial, this investigation considers: Descartes' Rule of Signs, the Budan-Fourier Theorem, and versions of Sturm's Method in contrast to the approximate root count gleaned from graphing utilities. Online applets are provided to allow the reader to freely experiment with different polynomial examples. Additionally, activities at the end of each section facilitate further investigations and deeper understanding of the topics.

Keywords: Descartes, Real Roots, Polynomials, Sturm

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While technology provides the opportunities for countless wonderful mathematical investigations, explications, and applications, technology can also hide – often within its sophisticated coding – beautiful, valuable, and powerful historic mathematics. At times, some technological tools (e.g., graphing) can minimize other investigations. For instance, when asked to count the real roots a particular real polynomial possesses, rather than employing algebraic techniques, a student may be more apt to simply use technology to graph the function, count *x*-intercepts, and account for odd and even multiplicities. However, even this technique may provide insufficient detail regarding the exact multiplicity of roots: for instance, $f(x) = (x-a)(x-b)^2$ and $g(x) = (x-a)^3(x-b)^4$ produce essentially the same *x*-axis behavior on the graph. Furthermore, depending on the window and zoom of a graphing technology, two distinct real roots in close proximity may be interpreted as a double real root or an extremum only slightly departed from the axis may be perceived as intersecting the *x*-axis. Thus, it can be argued that, in some cases, graphing techniques provide powerful, albeit often only approximate results. On the other hand, algebraic techniques – some even hundreds of years old – can produce exact results. It is valuable to understand the mathematics newly available through the power of technology and dynamic multiple representations along with the mathematics of the past.

In this paper, we investigate the enumeration of real roots of a real polynomial in the historical context preceding graphing technology and demonstrate that techniques still have significant value (Baker, 1892; Davies, 1845; Olney, 1885). We believe that providing students opportunity to investigate these ideas increases the richness of their mathematical experiences. For curious students, results like Descartes' Rule of Signs provide an accessible yet deep area of study that introduces them to a wonderful historical evolution of theorems regarding real and complex root counting associated with real polynomials. To enhance the reader's interactive experience, online applets are provided for the reader to experiment with different notions and polynomials. These applets, replicating centuries-old mathematics, are constructed in Sage and no software needs to be downloaded or installed for their use.

This brief investigation has a number of integrated purposes. First, in the context of the central role polynomials have in almost all mathematics, we hope to resurrect beautiful, historical mathematical ideas and consider the notion of exact versus approximate mathematical results. Second, we wish to present these mathematical ideas in a manner appropriate to as wide an audience as possible. We hope that advanced, high school mathematics students, college students, and their respective instructors will find this material interesting and engaging. The dynamic online applets allow for practice and deeper investigation, opening the topic to a wider audience who desire to see examples beyond those presented in the text. Since proofs of these mathematical ideas are relatively advanced, we provide references to some proof resources conveniently accessible online that are more readable than others. Third, since we hope that these materials can be used for deeper student investigations by either motivated students or in a study of the history of mathematics, we have added activities at the end of each section to facilitate further investigations of the topics.

The Problem

Given a polynomial with real coefficients, $p(x) = a_n x^n a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$, one may wonder how many real roots $p(x)$ possesses and exactly where they are located. While it is simple to locate the real roots when the polynomial is in factored form, when written in expanded form, root finding may be difficult or impossible to do exactly. Therefore, we resurrect a number of old mathematical techniques that can be applied to enumerating the number of real roots and determining their values in respect to polynomial.

Counting Real Roots

Descartes' Rule of Signs

Most mathematics students in grades 10-16 are informally and briefly acquainted with Descartes' Rule of Signs. This technique provides an upper bound on the number of positive real roots (counting multiplicities) and an upper bound on the number of negative real roots (counting multiplicities) of a polynomial. This interesting theorem is stated:

> Given a real polynomial, $p(x)$, ordered by descending variable exponent, then the number of:
>
> > **positive roots** (counting multiplicity) of $p(x)$ is either equal to the number of sign changes between consecutive nonzero coefficients, or is less than it by an even number.
> >
> > **negative roots** (counting multiplicity) is the number of sign changes of the coefficients of $p(-x)$ or less than it by an even number.
>
> Finally, it can be noted that zero is a root only when $p(x)$ has no constant term. In summary, the number of sign changes provides an upper bound for the number of positive and negative real roots respectively.

Example: Let

$$\begin{aligned} p(x) &= x^7 - 4x^6 - 4x^5 + 26x^4 - x^3 - 46x^2 + 4x + 24 \\ &= \underbrace{(x-3)(x-2)^2(x-1)(x+1)^2(x+2)}_{\text{for later verification}} \end{aligned}.$$

To find the number of positive roots, we note that there are four changes of sign, denoted (+-, --, -+, +-, --, -+, ++):

$$p(x) = \underbrace{x^7 - 4x^6}_{\text{sign change}} \underbrace{-4x^5 + 26x^4}_{\text{sign change}} \overbrace{- x^3 -46x^2}^{\text{sign change}} \underbrace{+ 4x}_{\text{sign change}} + 24 \cdot$$

To find the number of negative roots, we note that there are three changes of sign, denoted (--, -+, ++, ++, +-, --, -+):

$$p(-x) = -x^7 \underbrace{-4x^6 + 4x^5}_{\text{sign change}} + 26x^4 + \underbrace{x^3 - 46x^2}_{\text{sign change}} \underbrace{-4x + 24}_{\text{sign change}} .$$

Thus, while Descartes' Rule of Signs allows for the possibility of having four, two, or no positive roots and three or one negative root, our polynomial, in fact, has four positive and three negative roots (counting multiplicity).

The ease of implementation of this Rule comes at the cost of only providing bounds on, and not the exact number of, roots. Therefore, there are either four, two, or zero positive real roots and either three or one negative real roots (and, in this example, zero is not a root). While this enumeration of roots has value, it can be quickly seen that it also has inherent weaknesses in its provided choices for the number of roots.

Although an example is provided above regarding Descartes' Rule of Signs, readers may wish to experiment with numerous examples to ensure that the process is understood. To do so, use the online applet: Descartes' Rule of Signs. (A proof for Descartes' Rule of Signs can be found in Albert (1943).)

Budan-Fourier Theorem

Above, Descartes' Rule of Signs provided upper bounds for the number of positive or negative real roots (i.e., the intervals $I = (0, \infty)$ of positive real numbers and $(-\infty, 0)$ of negative real numbers). However, one may desire to count the number of real roots within other intervals. The Budan-Fourier Theorem gives an upper bound on the number of real roots (counting multiplicity) of a real polynomial in a given interval $I=(a,b]$, where $a=-\infty$ and/or $b=\infty$ are allowed. [Note that when $b=\infty$, $I=(a,b)$.] The process can be simply stated as:

- Begin with a (nonzero) real polynomial and compute all of its derivatives. Evaluate this list of derivatives at $x=a$.
- After ignoring all zeros, count the number of times the list of numbers switches from positive to negative numbers or vice-versa.
- Next, do the same for $x=b$.
- The number of sign changes at $x=a$ minus the number of sign changes at $x=b$ gives an upper bound on the number of real roots of that polynomial (counting multiplicity). In fact, if the difference in sign changes is *s*, then this polynomial has $s-2k$ real roots (counting multiplicity) for some positive integer *k*. For example, if $s=3$, then there are either one or three real roots in our interval. On the other hand, if $s=4$, then there are either four, two, or no real roots in our interval.

Descartes' Rule of Signs is a special case of the Budan-Fourier Theorem. If $a=0$, the sequence of sign changes in $f(0), f'(0), f''(0), \cdots$ is the same as the sign changes in the coefficients of $f(x)$. If $b=\infty$, then all of signs of $f(\infty), f'(\infty), f''(\infty), \cdots$ are the same. [By $f(\pm\infty)$ we mean the limit of $f(x)$ as *x* approaches $\pm\infty$. The value of $f(+\infty)$ is the sign of the leading coefficient times infinity if $f(x)$ is non-constant and is $f(x)$ itself if $f(x)$ is constant. On the other hand, $f(-\infty)$ is $(-1)^{\text{degree}(f)}$ times the sign of the leading coefficient times infinity if $f(x)$ is non-constant and just $f(x)$ itself if $f(x)$ is constant.] Thus, Descartes' Rule of Signs counts the same thing as Budan-Fourier for $a=0$ and $b=\infty$, that is the roots in $I=(0,\infty)$ (i.e. the positive real roots).

Extending Descartes Rule, the Budan-Fouier method can be used to search for the locations of roots. By examining Budan-Fouier's results for different values of *a* and *b*, one can often find regions where there are no roots and regions where there must be a root. Of course, this method is still limited in that it provides a bound for root counts rather than an exact root count. (A proof for the Budan-Fourier Theorem can be found in Conkwright (1943).)

To experiment with numerous examples to ensure that the process is understood, the reader may use the online applet: Budan-Fourier Theorem.

Activity:

1. Without using the applet provided, use the Budan-Fourier theorem to count the real roots in $p(x) = x^6 + 2x^5 - 4x^4 - 8x^3$.
2. Use the online applet to confirm your findings.
3. Use Descartes Rules of Signs to determine the number of positive and negative real roots. (One positive root and two or zero negative roots.) Compare your results with that from #1. Explain whether or not the two answers agree.
4. Use the online applet to confirm your findings.
5. Consider the following to confirm your findings:

$$\begin{aligned} p(x) &= x^6 + 2x^5 - 4x^4 - 8x^3 \\ &= (x+2)^2 x^3 (x-2) \end{aligned}.$$

Sturm's Method

The previous techniques provided only upper bounds on root counts. This weakness is overcome using Sturm sequences, through which an exact number of distinct real roots can be determined. (A brief discussion and proof are provided in Fisher (1999, p. 386-387).) Notably, Sturm sequences are determinable using only the simple operations of taking the first derivative of a real polynomial and polynomial division. To best understand Sturm's Method, we begin with the Euclidean Algorithm for determining the greatest common divisor of two natural numbers and the Euclidean Algorithm for polynomials. Since Sturm's Method uses a modification of these Algorithms, investigating these Algorithms provides scaffolding for following discussions.

Euclidean Algorithm. For millennia, mathematicians have employed the Euclidean Algorithm to determine the greatest common divisor of two natural numbers. The Algorithm is based on the following simple observation: If $m = nq + r$, then the common divisors of m and n are exactly the same as the common divisors of n and r. Let $\gcd(m, n)$ denote the greatest common divisor of m and n. This means that $\gcd(m, n) = \gcd(n, r)$. Each time the division is performed and a remainder determined, the old argument can be exchanged for a smaller new one (i.e. swap out m for r). Since the remainders continue to diminish, eventually one of them must become 0. At this point, $\gcd(r, 0) = r$ and so the last nonzero remainder is the $\gcd(m, n)$.

Many online applets use the Euclidean Algorithm in order to calculate the greatest common divisor of two inputted values. However, in so doing, the Algorithm becomes hidden. To further investigate the Euclidean Algorithm and witness examples worked out, the reader is invited to use the applet at Euclidean Algorithm.

Euclidean Algorithm for Polynomials. It is often necessary to seek for the greatest common divisor of two or more polynomials. Finding the greatest common divisor of two polynomials, $m(x)$ and $n(x)$, can be accomplished in the exact same manner as with two natural numbers. Again, it the case that, if $m(x)$ is divided by $n(x)$ and thereby $m(x) = n(x)q(x) + r(x)$, then the common divisors of $m(x)$ and $n(x)$ are exactly the common divisors of $n(x)$ and $r(x)$ so that the $\gcd(m(x), n(x)) = \gcd(n(x), r(x))$. Continuing the process of dividing polynomials and each time swapping out one of the polynomials with a remainder, leads to each remainder having a lower degree until $\gcd(m(x), n(x)) = \cdots = \gcd(r(x), 0) = r(x)$, where $\boldsymbol{r(x)}$ is the last non-zero remainder.

To explore examples of determining the greatest common divisor of two polynomials, consider the online applet at Polynomial Euclidean Algorithm. It may be beneficial to initially enter polynomials in factored form in order to best observe the results.

Activity:

1. Let $m(x) = 2x^5 + 3x^4 + 5x^2 + 3x + 2$ and $n(x) = x^4 + x^3 - x^2 + 2x$.
2. Perform the Euclidean Algorithm for polynomials to determine the $\gcd(m(x), n(x))$.
3. Use the online applet to confirm your results.
4. Descartes' Rule of Signs reveals that both $m(x)$ and $n(x)$ have at least one real root. In fact, it says that $n(x)$ has exactly one real root. Budan-Fourier could, for example, tell us that both of these polynomials have a root in the interval $(-5,1]$ (use $a = -5$ and $b = 1$). The Euclidean Algorithm reveals that they SHARE a root at $x = -2$.
5. Confirm the findings in #4.

Returning to Sturm's Method

Sturm's method for counting distinct real roots of a real polynomial restricted to an interval begins with the computation of a sequence of polynomials known as a Sturm sequence or Sturm chain.

- Begin with polynomial $p_0(x) = p(x)$ and compute its derivative $p_1(x) = p'(x)$.
- Then determine $p_0(x)/p_1(x)$. This yields a polynomial quotient and remainder.
- Let $p_2(x) = -\left(remainder(p_0(x)/p_1(x))\right)$.
- Now continue dividing $p_i(x)$ by $p_{i+1}(x)$and setting p_{i+1} equal to the negative remainder until a zero remainder is achieved. This will inevitably occur since each division yields a negative remainder of lower degree.

The Sturm sequence, denoted the canonical Sturm chain, can be depicted as:

$$
\begin{aligned}
p_0(x) &:= p(x), \\
p_1(x) &:= p'(x), \\
p_2(x) &:= -\big(rem\big(p_0/p_1\big)\big) = p_1(x)q_0(x) - p_0(x), \\
p_3(x) &:= -\big(rem\big(p_1/p_2\big)\big) = p_2(x)q_1(x) - p_1(x), \\
&\vdots \\
p_{m+1}(x) &:= -\big(rem\big(p_{m-1}/p_m\big)\big) = 0
\end{aligned}
$$

Computing the canonical Sturm chain is almost the same thing as running the Euclidean Algorithm on $\boldsymbol{p(x)}$ and its derivative $\boldsymbol{p'(x)}$. The only difference is that at each stage we replace a previous polynomial with a negative remainder instead of a remainder. Now constant multiples have no effect on the greatest common divisor of polynomials so it still the case that the Sturm sequence will compute the greatest common divisor of $\boldsymbol{p(x)}$ and $\boldsymbol{p'(x)}$.

With Sturm's sequence of polynomials defined, we recognize that an anomaly occurs when the initial polynomial has a repeated root at either $x = a$ or $x = b$ on the interval [*a*, *b*]. Thus, we consider two cases: Case 1, when neither *a* nor *b* is a repeated root and Case 2, when either *a* or *b* is a repeated root.

Case 1. Let $p_0(x), p_1(x),\ p_2(x),\ \cdots, p_m(x)$ be the Sturm sequence for $p(x) = p_0(x)$. For any real number *t* or $t = \pm\infty$, let $\boldsymbol{\sigma(t)}$ denote the number of sign changes in the sequence $p_0(t), p_1(t), p_2(t), \cdots, p_m(t)$ [as with Descartes – we ignore zeros.]. Then for any two real numbers $\boldsymbol{a < b}$ (we also allow $\boldsymbol{a = -\infty}$ and/or $\boldsymbol{b = \infty}$), as long as neither *a* nor *b* is a repeated root, $\boldsymbol{\sigma(a) - \sigma(b)}$ is the number of distinct real roots lying in the interval $\boldsymbol{I = (a,b]}$ or $\boldsymbol{I = (a,\infty)}$ if $\boldsymbol{b = \infty}$.

We now consider the second case, when either $\boldsymbol{x = a}$ or $\boldsymbol{x = b}$ is a repeated root. Evaluating the Sturm sequence of polynomials at a repeated root will yield an unusable string of 0's. But all is not lost, if the process is modified. First, determine the last nonzero term of the sequence. Dividing the entire sequence by this term then modifies the canonical Sturm chain and eliminates the unwanted string of zeros. This modified Sturm sequence is useable even when we evaluate at repeated roots.

It is an interesting and relatively unfamiliar fact that $\gcd\left(p(x), p'(x)\right)$ is actually the product of all of repeated copies of the factors in $p(x)$. This means that $p_m(x)$ (the last term in the Sturm sequence) is just repeated copies of factors of $p(x)$. Then $p(x)/p_m(x)$ has all of the same roots as $p(x)$ [real and complex] but $p(x)/p_m(x)$ has no repeated roots.

Case 2. If we replace the Sturm sequence $p_0(x), p_1(x),\ p_2(x),\ \cdots, p_m(x)$ with a modified sequence $\frac{p_0(x)}{p_m(x)}, \frac{p_1(x)}{p_m(x)}, \frac{p_2(x)}{p_m(x)}, \cdots, \frac{p_m(x)}{p_m(x)} = 1$, then Sturm's method still applies, but now we do not need the caveat about $x = a$ or *b* being a repeated root. For the modified sequence $\sigma(a) - \sigma(b)$ always counts the number of distinct real roots in $I = (a,b]$.

The theorem then leads to the process: Evaluate $p_0(x), p_1(x),\ p_2(x),\ \cdots, p_{i+1}(x)$ at $x = a$, yielding a list of real numbers. Count the number of sign changes in this list (ignoring zeros) and call the number of sign changes *A*. Then do the same for $x = b$ and call the number of sign changes *B*. If neither $x = a$ nor $x = b$ is a repeated root, the number of distinct roots in the interval $I = (a,b]$ is exactly $A - B$.

Notably, while the modified Sturm sequence can always be used without exception, if the polynomial case does not possess repeated roots at the boundaries of the interval in question, the regular, canonical Sturm sequence is easier to compute (avoiding extra polynomial divisions).

However, when the interval in question is $I = (-\infty, \infty)$ since $\pm\infty$ are not roots, the original, unmodified Sturm sequence always works and counts all distinct real roots.

Example: Let $p_0(x) = p(x) = (x-1)^2(x+2) = x^3 - 3x + 2$. Let us now compute the first derivative: $p_1(x) = p'(x) = 3x^2 - 3$. Next, polynomial division yields $p(x) = p'(x) \cdot \frac{1}{3}x + (-2x + 2)$. Thus,

$$p_2(x) = -rem\big(p_0(x), p_1(x)\big) = -rem\big(p(x), p'(x)\big) = -(-2x+2) = 2x - 2.$$

Finally, $p_1(x) = p_2(x) \cdot \left(\frac{3}{2}x + \frac{3}{2}\right) + 0$ so that $p_3 = -rem\big(p_1(x), p_2(x)\big) = 0$. Thus, our sequence ends with $p_2(x)$.

Let us now evaluate our Sturm sequence at $a = 0$ and $b = 2$ (Case 1).

$$p_0(0) = 2 \qquad p_1(0) = -3 \qquad p_2(0) = -2$$

$$p_0(2) = 4 \qquad p_1(2) = 9 \qquad p_2(2) = 2$$

Thus, we have one sign change at $a = 0$ and no sign changes at $b = 2$, so $\sigma(a) = \sigma(0) = 1$ and $\sigma(b) = \sigma(2) = 0$. Therefore, $p(x)$ has $\sigma(a) - \sigma(b) = 1 - 0$ root in $I = (0{,}2]$.

If we wished to apply this method at $a = 0$ and $b = 1$ (Case 2), we run into a problem: $p_0(1) = p_1(1) = p_2(1) = 0$. Here we need the modified Sturm sequence, since $b = 1$ is a repeated root. Dividing by $p_2(x) = 2x - 2$, we get:

$$r_0(x) = \frac{p_0(x)}{p_2(x)} = \frac{1}{2}x^2 + \frac{1}{2}x - 1 \qquad r_1(x) = \frac{p_1(x)}{p_2(x)} = \frac{3}{2}x + \frac{3}{2} \qquad r_2(x) = \frac{p_2(x)}{p_2(x)} = 1.$$

Now, $r_0(0) = -1$, $r_1(0) = \frac{3}{2}$, and $r_2(0) = 1$ (i.e., $\sigma(a) = \sigma(0) = 1$ sign change) and $r_0(1) = 0$, $r_1(1) = 3$, and $r_2(1) = 1$ (i.e., $\sigma(b) = \sigma(1) = 0$ sign changes). Therefore, there is $\sigma(a) - \sigma(b) = 1 - 0 = 1$ root in $I = (0{,}1]$.

An online applet is provided that allows the reader to experiment with any number of examples of Sturm's Method, see these examples worked out, and come to better understand the process of using these sequences. This applet can be found at Sturms' Method.

Activity:

1. Let $p(x)=x^3-3x+3$. Confirm that Sturm's method reveals that $p(x)$ has one real root.
2. Let $q(x)=x^3-3x+2$. Confirm that Sturm's method reveals that $q(x)$ has two real roots.
3. Let $r(x)=x^3-3x+1$. Confirm that Sturm's method reveals that $r(x)$ has three real roots.
4. Use the advanced Sturm sequence applet (provided in the following section) to reveal what the multiplicities are.
5. Let $s(x)=x^4-x^3-2x^2+3x-1$. Sturm's method tells us that there are three distinct real roots and the advanced Sturm's method tells us one root is repeated). Note that if $s(x)$ is graphed on the interval $(-2,2)$, it looks like there are only two roots!

Extending Sturm's Method

While Sturm's Method counts the number of distinct real roots of a real polynomial in an interval, it cannot determine the multiplicity of each root. Employing an iterative use of Sturm's Method can determine the number of real roots in an interval, the multiplicity of these counted roots, and the number of complex roots. In particular, since the $\gcd\left(p(x),p'(x)\right)=p_m(x)=q(x)$ (the last term in the Sturm sequence) contains any repeated copies of factors of $p(x)$, one can apply Sturm's Method to $q(x)$ to find the number of distinct real roots in an interval that are repeated at least twice. Applying Sturm's Method to the last polynomial in $q(x)$'s sequence will find number of distinct real roots in an interval that are repeated at least three times. Continuing in this fashion eventually reveals all how many distinct real roots in an interval of each multiplicity there must be. We refer to this repeated use of Sturm's original method as the Advanced Sturm's Method. If we use $I=(-\infty,\infty)$, we have found ***ALL*** real roots counting multiplicity, so any remaining root is complex.

With additional application and modification of the Advanced Sturm's Method, techniques can be used to determine ranges in which the real roots exist, thereby assisting to determine the values and multiplicities of these real roots. To investigate these ideas, readers are provided both an online applet and the following activity. (See Advanced Sturm's Method.)

Activity:

1. Open the Advanced Sturm's Method applet. Deselect "Show table?" and "Show plot?".
2. For $p(x)$, enter $x^{10}-2x^9-8x^8+18x^7+15x^6-48x^5+8x^4+32x^3-16x^2$ and press "Update".
3. Note that the results are:

   $p(x)$ has 10 real roots counting multiplicity.

   There is 1 real root of multiplicity 3.

   There are three real roots of multiplicity 2.

   There is 1 real root of multiplicity 1.

   $p(x)$ has no complex roots.
4. Compare these results with that of Descartes' Rule of Signs which reports that $p(x)$ would have five, three, or one positive root and three or one negative roots. Note that

the results from Sturm and Descartes are consistent, albeit both providing slightly different information. This will later be investigated in more detail.

5. Select “Show table?” and enter and enter a list of integers ranging from -5 to 5 into “Table values”. [This can be entered as [-5,-4,-3,-2,-1,0,1,2,3,4,5] or range(-5,6).] Press “Update”. Note that five changes in signs alter to four changes in sign as x goes from -3 to -2. This indicates that there must be a real root in the interval $(-3,-2]$. Similarly, real roots must exist in the intervals $(-2,-1]$, $(-1,0]$, $(0,1]$, and $(1,2]$. Additionally, no roots exceed $x=2$. Therefore, all roots are in the interval $(-3,2]$ Notably, this information far surpasses information available via Descartes’ Rule of Signs.

6. For confirmation of all results, note that

$$\begin{aligned} p(x) &= x^{10} - 2x^9 - 8x^8 + 18x^7 + 15x^6 - 48x^5 + 8x^4 + 32x^3 - 16x^2 \\ &= \underbrace{(x+2)^2(x+1)}_{\substack{\text{negative real roots} \\ x=-2,-1}} x^2 \underbrace{(x-1)^3(x-2)^2}_{\substack{\text{positive real roots} \\ x=1,2}} \end{aligned}$$

7. Repeat the previous activity (parts 1-6) with the polynomial

$$p(x) = x^{10} - 5x^9 - x^8 + 41x^7 - 49x^6 - 87x^5 + 177x^4 - 9x^3 - 128x^2 + 60x$$

Note that Descartes’ Rule of Signs would report this polynomial to have six, four, two, or zero positive real roots and three or one negative real roots.

⋮

12. For confirmation of all results, note that

$$\begin{aligned} p(x) &= x^{10} - 5x^9 - x^8 + 41x^7 - 49x^6 - 87x^5 + 177x^4 - 9x^3 - 128x^2 + 60x \\ &= \underbrace{(x+2)^2(x+1)}_{\substack{\text{negative real roots} \\ x=-2,-1}} x \underbrace{(x-1)^3(x-3)}_{\substack{\text{positive real roots} \\ x=1,3}} \underbrace{\left((x-2)^2+1\right)}_{\substack{\text{complex root pair} \\ 2\pm i}} \end{aligned}$$

Culminating activity:

Use all of the tools previously mentioned and find the value and multiplicity of all real roots of $p(x) = x^{10} + x^9 - 19x^8 - 15x^7 + 149x^6 + 37x^5 - 545x^4 - 33x^3 + 714x^2 + 10x - 300$ and the number of nonreal complex roots. (Note that this is not asking for the number of roots, but the actual value and multiplicity of the roots.)

Summary and Conclusion

We have seen that the Advanced Sturm's Method can be used to both count the number of real roots (and their multiplicities) in an interval and help locate these real roots. This investigation provides a beginning for further study. There are numerous intriguing theorems awaiting those wishing to delve more deeply into these matters. For instance, consider the theorem:

Let $p(x)$ be a real polynomial, *m* and *l* real numbers, and *m*>0 and *l*<0. Let

$$\frac{p(x)}{x-m} = q_m(x) + \frac{r_m(x)}{x-m} \text{ and } \frac{p(x)}{x-l} = q_l(x) + \frac{r_l(x)}{x-l}.$$

If all coefficients of $q_m(x)$ and $r_m(x)$ are of the same sign, then $p(x) = 0$ has no root greater than *m*. [0 may be denoted 0 or -0.] If all coefficients of $q_l(x)$ and $r_l(x)$ alternate signs, then $p(x) = 0$ has no root less than *l*.

While graphing technologies are wonderful tools, before the existence of graphing technologies, beautiful, historical mathematics abounded. Investigating some of theorems can promote even greater student inquiry and intrigue. Enjoy!

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Resources

The full URLs for the hyperlinks provided in the paper are listed below.

http://mathsci2.appstate.edu/~cookwj/sage/algebra/Descartes_rule_of_signs.html

http://mathsci2.appstate.edu/~cookwj/sage/algebra/Budan-Fourier-theorem.html

http://mathsci2.appstate.edu/~cookwj/sage/algebra/Euclidean_algorithm.html

http://mathsci2.appstate.edu/~cookwj/sage/algebra/Euclidean_algorithm-poly.html

http://mathsci2.appstate.edu/~cookwj/sage/algebra/Sturms_method.html

http://mathsci2.appstate.edu/~cookwj/sage/algebra/Sturms_method-advanced.html