Leibniz Algebras with Low Dimensional Maximal Lie Quotients

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Abstract

Every Leibniz algebra has a maximal homomorphic image that is a Lie algebra. In this paper, we classify cyclic Leibniz algebras over an arbitrary field. Such algebras have the 1-dimensional abelian Lie algebra as their maximal Lie quotient. We then give examples of Leibniz algebras whose associated maximal Lie quotients exhaust all 2-dimensional possibilities.[1]

1 Introduction

The theory of Leibniz algebras has blossomed since the pioneering work of Loday [L]. Transitioning from Lie to Leibniz algebras is similar to transitioning from commutative to non-commutative rings. Both transitions drop one defining property, leading to many new and interesting structures. In a Leibniz algebra we keep a version of the Jacobi identity but no longer assume that multiplication is alternating and hence not necessarily skew-symmetric either. To truly understand an algebraic structure one needs a varied collection of illuminating examples. In this paper we seek to provide a small collection of examples of non-Lie (left) Leibniz algebras.

In [SS] the authors provide a classification of cyclic Leibniz algebras over the complex field. We offer a variant proof which avoids the use of n^{th}-roots and thus provides a complete classification of cyclic Leibniz algebras over arbitrary fields. In addition, we construct two classes of non-cyclic Leibniz algebras with non-isomorphic 2-dimensional maximal Lie quotients, exhausting all possibilities for such quotients.

The paper is structured as follows: after providing some background in Section 2, we use Section 3 to construct and classify all cyclic Leibniz algebras over an arbitrary field. The next two sections present examples of Leibniz algebras with both non-abelian (Section 4) and abelian (Section 5) 2-dimensional maximal Lie quotients.

2 Background

Let $\mathbb{F}$ be a field. For our purposes it suffices to consider only finite dimensional vector spaces over $\mathbb{F}$.

Definition 2.1. *Let L be a vector space equipped with a bilinear map $[\,\cdot\,,\,\cdot\,] : L \times L \to L$, called a* bracket*, such that for all $x, y, z \in L$ the (left)* Leibniz identity*: $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$ holds. Then L is called a (left)* Leibniz algebra*.*

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Briefly, a (left) Leibniz algebra is an algebra whose left multiplication operators are derivations. Similarly we could assume that right multiplication operators are derivations and define the notion of a right Leibniz algebra. Just as with many other algebraic constructions our choice of left versus right is arbitrary. All of our results for left Leibniz algebras can easily be translated to results for right Leibniz algebras. For the remainder of the paper Leibniz algebra will mean left Leibniz algebra.

Notice that the Leibniz identity could replace the Jacobi identity in the definition of a Lie algebra. In fact, the left Leibniz identity, the corresponding right Leibniz identity: $[[y, z], x] = [y, [z, x]] + [[y, x], z]$, and the Jacobi identity: $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ are all equivalent if we assume our bracket is bilinear and alternating: $[x, x] = 0$ for all x. We refer the reader to [DMS] for more details concerning basic definitions related to Leibniz algebras.

Definition 2.2. *For L a Leibniz algebra,* $\mathrm{Leib}\,(L) = \mathrm{span}_{\mathbb{F}}\{[x, x] \mid x \in L\}$.

We have that L is a Lie algebra if and only if $\mathrm{Leib}\,(L) = \{0\}$. Notice that $\mathrm{Leib}\,(L)$ is a (two-sided) ideal of L. Moreover, $L/\mathrm{Leib}\,(L)$ is the largest quotient of L that is a Lie algebra. Specifically, if I is any ideal of L such that L/I is a Lie algebra, then $\mathrm{Leib}\,(L) \subseteq I$. Here we use the term *ideal* in the familiar Lie algebra sense: a subalgebra I of a Leibniz algebra L is a (two-sided) ideal of L if and only if $[L, I]$ and $[I, L]$ are both contained in I. We write $I \lhd L$ when I is a ideal of L.

Many other definitions extend directly from Lie to Leibniz algebras. As a second example, we say L is an *abelian* Leibniz algebra if and only if $[L, L] = \{0\}$, that is if $[x, y] = 0$ for all $x, y \in L$. The definitions of nilpotency and solvability also carry over without modification.

Definition 2.3. *Recall that $L^1 = L$ and $L^{j+1} = [L, L^j]$ for $j \geq 1$ gives us the* lower central series*. L is* nilpotent *of class n if $L^{n+1} = \{0\}$ but $L^n \neq \{0\}$. In particular, L is* nilpotent *if $L^n = \{0\}$ for some $n \geq 1$. Likewise, $L^{(0)} = L$ and $L^{(j+1)} = [L^{(j)}, L^{(j)}]$ for $j \geq 0$ gives us the* derived series*. L is* solvable *if $L^{(n)} = \{0\}$ for some $n \geq 0$.*

The proofs of many basic results given in introductory Lie algebra texts such as [EW] apply just as well to Leibniz algebras. In particular, abelian implies nilpotent and nilpotent implies solvable. Recall that $\mathrm{rad}(L)$ is the largest solvable ideal of L. As with Lie algebras, this is just the sum of all ideals I of L such that I itself is a solvable algebra. Likewise, $\mathrm{nil}(L)$ is the largest nilpotent ideal.

The notion of internal direct sum for Leibniz algebras also carries over from Lie theory. As with Lie algebras, if $L = L_1 \oplus \cdots \oplus L_n$ is an internal direct sum of Leibniz algebras, each L_i is in fact an ideal of L and L is isomorphic to the external direct sum of Leibniz algebras $L_1, \ldots, L_n$, defined in the obvious way.

Definition 2.4. *Let L be a Leibniz algebra with subalgebras $L_1, \ldots, L_n$. We write $L = L_1 \oplus \cdots \oplus L_n$, an internal direct sum of Leibniz algebras, if $L = L_1 \oplus \cdots \oplus L_n$ as subspaces and $[x, y] = 0$ for any $x \in L_i$ and $y \in L_j$ where $i \neq j$.*

It is not hard to show that for $I_j \lhd L_j$, we have $(L_1 \oplus \cdots \oplus L_n)/(I_1 \oplus \cdots \oplus I_n) \cong (L_1/I_1) \oplus \cdots \oplus (L_n/I_n)$ with the direct sum on the right an external direct sum. Likewise, $Z(L_1 \oplus \cdots \oplus L_n) = Z(L_1) \oplus \cdots \oplus Z(L_n)$, $\mathrm{Leib}\,(L_1 \oplus \cdots \oplus L_n) = \mathrm{Leib}\,(L_1) \oplus \cdots \oplus \mathrm{Leib}\,(L_n)$, and $[L_1 \oplus \cdots \oplus L_n, L_1 \oplus \cdots \oplus L_n] = [L_1, L_1] \oplus \cdots \oplus [L_n, L_n]$.

Some important definitions from Lie theory require minor modifications as we move to Leibniz algebras. For example, if we apply the Lie theory definitions of simple and semisimple algebras directly to Leibniz algebras, both simple and semisimple Leibniz algebra would necessarily be Lie and thus there would be nothing new to consider. We modify these definitions for Leibniz algebras as follows:

Definition 2.5. *Let L be a Leibniz algebra. L is* simple *if and only if $[L,L] \neq \mathrm{Leib}(L)$ and $\{0\}$, $\mathrm{Leib}(L)$, and L are the only ideals of L. L is* semisimple *if and only if* $\mathrm{rad}(L) = \mathrm{Leib}(L)$.

When L is also a Lie algebra, $\mathrm{Leib}(L) = \{0\}$, so these definitions collapse back down to the usual definitions for a Lie algebra. In fact, these definitions guarantee that L is simple (resp. semisimple) as a Leibniz algebra if and only if $L/\mathrm{Leib}(L)$ is simple (resp. semisimple) as a Lie algebra.

When working with Lie algebras, taking powers of elements is uninteresting: $x^1 = x$ and then $x^2 = [x,x] = 0$ because of the alternating axiom. In Leibniz algebras much more is possible. We fix the notation $x^1 = x$, $x^2 = [x,x]$, and in general, $x^{n+1} = [x, x^n]$ for $n \geq 1$. Consider the following basic, well-known result:

Lemma 2.6. *Let L be a Leibniz algebra and $x, y \in L$. Then $[[x,x],y] = 0$ and more generally $[x^n, y] = 0$ for all $n \geq 2$. Moreover, the only potentially non-zero n^{th}-power of x is* $x^n = \underbrace{[x,[x,\ldots,[x,x]\cdots]]}_{n-\text{times}}$.

Proof: The Leibniz identity states that $[x,[x,y]] = [[x,x],y] + [x,[x,y]]$ so that $0 = [[x,x],y]$. Assume inductively that $[x^n, z] = 0$ for any $z \in L$ and some $n \geq 2$. The Leibniz identity states that $[x,[x^n,y]] = [[x,x^n],y] + [x^n,[x,y]]$. By our inductive hypothesis, we have $[x,0] = [x^{n+1},y] + 0$ so that $[x^{n+1},y] = 0$.

Finally, the only first and second powers of x are $x^1 = x$ and $x^2 = [x,x]$. Third powers of x can be written either as x^3 or $[[x,x],x] = 0$. Assume that all k^{th}-powers of x other than x^k are 0 where $1 \leq k < n$ and let w be some n^{th}-power of x. Then $w = [u,v]$ where u and v are k^{th} and ℓ^{th}-powers of x such that $k + \ell = n$. By induction, if $u \neq 0$ and $v \neq 0$, we must have $u = x^k$ and $v = x^\ell$. So either $k \geq 2$ and thus $w = [u,v] = [x^k, v] = 0$ or $k = 1$ and we have $w = [u,v] = [x, x^\ell] = x^{\ell+1} = x^n$. $\square$

We can see that generally Leibniz algebras are not power associative. Notice that for a right Leibniz algebra we would have that the only potentially non-zero powers would be of the form $[[\cdots[x,x],\ldots,x],x]$. This means that if an algebra was both a left and right Leibniz algebra, the only non-zero power could be $x^2 = [x,x]$. In fact, $L = \mathrm{span}_{\mathbb{F}}\{x, x^2\}$ where $[x,x] = x^2$, $[x,x^2] = [x^2,x] = [x^2,x^2] = 0$ gives an example of a simultaneously left and right Leibniz algebra which is not a Lie algebra.

3 Cyclic Leibniz Algebras

A cyclic Leibniz algebra is a Leibniz algebra that can be generated from a single element. We do not consider cyclic Lie algebras since the only cyclic Lie algebras are either the trivial algebra $\{0\}$ or the 1-dimensional abelian Lie algebra. Scofield and Sullivan [SS] have classified complex cyclic Leibniz algebras. In this section, we give a similar construction which allows us to classify cyclic (left) Leibniz algebras over an arbitrary field.

Definition 3.1. *Let L be a Leibniz algebra. L is* cyclic *if and only if there exists some $x \in L$ such that $L = \langle x \rangle = \text{span}_{\mathbb{F}}\{x^k \mid k = 1, 2, \ldots\}$. If $L = \langle x \rangle$, we call x a* generator *of L.*

The trivial algebra $\{0\} = \langle 0 \rangle$ is cyclic. Likewise, any 1-dimensional algebra is cyclic as it is generated by any non-zero element.

Let $L \neq \{0\}$ be a cyclic (left) Leibniz algebra and fix a generator $x \neq 0$. By definition $L = \langle x \rangle = \{x^k \mid k = 1, 2, \ldots\}$ and since L is finite dimensional, we must have that $\{x, x^2, \cdots, x^{n+1}\}$ is linearly dependent for some $n \geq 1$. Let n be the smallest such power. This means that $\{x, x^2, \ldots, x^n\}$ is linearly independent and x^{n+1} can be written as a linear combination of $\{x, \ldots, x^n\}$. Consequently all higher powers of x can be written as a linear combination of $x, x^2, \ldots, x^n$. Thus $\beta = \{x, x^2, \ldots, x^n\}$ is a basis for L and so $\dim(L) = n$.

We have $x^{n+1} \in L = \langle x \rangle = \text{span}_{\mathbb{F}}\{x, x^2, \ldots, x^n\}$. Let $x^{n+1} = \sum\limits_{i=1}^{n} c_i x^i$ where $c_i \in \mathbb{F}$. When $\dim(L) = n > 1$, Lemma 2.6 guarantees $0 = [x, 0] = [x, [x^n, x]]$. Applying the Leibniz identity and Lemma 2.6 once more yields

$$0 = [x, [x^n, x]] = [[x, x^n], x] + [x^n, x^2] = [x^{n+1}, x] + 0 = c_1 x^2 + \sum_{i=2}^{n} c_i [x^i, x] = c_1 x^2.$$

Since $\dim(L) = n > 1$, we conclude $x^2 \neq 0$ and thus $c_1 = 0$. Therefore, $x^{n+1} = \sum\limits_{i=2}^{n} c_i x^i$, a summation that does not involve $i = 1$.

It turns out that the necessary condition that $x^{n+1} = \sum\limits_{i=2}^{n} c_i x^i$ for some $c_2, \ldots, c_n \in \mathbb{F}$ is also sufficient for any n-dimensional cyclic Leibniz algebra $L = \langle x \rangle$.

Proposition 3.2. *Fix $n \geq 1$ and $c_2, \ldots, c_n \in \mathbb{F}$ and let $L = \text{span}_{\mathbb{F}}\{x, x^2, \ldots, x^n\}$ be an n-dimensional vector space. Define a bilinear operation on the basis $\{x, x^2, \ldots, x^n\}$ as follows: $[x, x^j] = x^{j+1}$ for $1 \leq j < n$, $[x, x^n] = \sum\limits_{i=2}^{n} c_i x^i$, and $[x^k, x^\ell] = 0$ for $k \geq 2$, $1 \leq \ell \leq n$. Then $L = \langle x \rangle$ is a cyclic Leibniz algebra.*

Proof: Clearly L is a cyclic algebra equipped with a bilinear operation. It just remains to verify the Leibniz identity. It is enough to do so on our basis. We note that when $n = 1$, $x^{n+1} = x^2 = 0$ and the Leibniz identity is $[x, [x, x]] = [x, 0] = 0 = 0 + 0 = [0, x] + [x, 0] = [[x, x], x] + [x, [x, x]]$. Assume $n > 1$ and let $1 \leq i, j, k \leq n$.

If $i \geq 2$, then

$$[x^i, [x^j, x^k]] = 0 = 0 + 0 = [0, x^k] + [x^j, 0] = [[x^i, x^j], x^k] + [x^j, [x^i, x^k]].$$

If $i = 1$ and $j = 1$, then

$$[x, [x, x^k]] = 0 + [x, [x, x^k]] = [x^2, x^k] + [x, [x, x^k]] = [[x, x], x^k] + [x, [x, x^k]].$$

If $i = 1$ and $2 \leq j < n$, then

$$[x, [x^j, x^k]] = [x, 0] = 0 = 0 + 0 = [x^{j+1}, x^k] + [x^j, x^{k+1}] = [[x, x^j], x^k] + [x^j, [x, x^k]].$$

If $i = 1$ and $j = n > 1$, then

$$[x, [x^n, x^k]] = [x, 0] = 0 = \sum_{m=2}^{n} c_m [x^m, x^k] = [x^{n+1}, x^k] + 0 = [[x, x^n], x^k] + [x^n, [x, x^k]].$$

Notice that here we used the fact that our sum begins at $m = 2$ so $[x^m, x^k] = 0$. $\square$

For $n > 0$ fix a cyclic Leibniz algebra L with basis $\beta = \{x, x^2, \ldots, x^n\}$. Next, we will further investigate the structure of this algebra by considering $\mathrm{Leib}(L)$ and the derived series of L. Note that by definition $x^2 \in \mathrm{Leib}(L)$. But then since $\mathrm{Leib}(L)$ is an ideal of L, $x^j \in \mathrm{Leib}(L)$ for all $j \geq 2$. Since brackets among elements of L never result in an element involving x itself, we conclude $\mathrm{Leib}(L) = \mathrm{span}\left\{x^2, x^3, \cdots, x^n\right\} = [L, L]$, an abelian Leibniz algebra of dimension $n - 1$. It quickly follows that the derived series for L is given by

$$L^{(0)} = L \supsetneq L^{(1)} = [L, L] = \mathrm{span}\left\{x^2, x^3, \cdots, x^n\right\} \supsetneq L^{(2)} = \{0\}.$$

The series goes to zero and thus cyclic Leibniz algebras are always solvable.

We next consider the lower central series of the cyclic Leibniz algebra $L = \langle x \rangle$ with basis $\beta = \{x, x^2, \ldots, x^n\}$ and $x^{n+1} = \sum\limits_{i=2}^{n} c_i x^i$. First consider the case when $x^{n+1} = 0$, that is when $c_2 = c_3 = \cdots = c_n = 0$. Then keeping in mind that only left multiplication by x can yield a nonzero result, we have $[L, \mathrm{span}\{x^m, x^{m+1}, \ldots, x^n\}] = \mathrm{span}\{[x, x^m], [x, x^{m+1}], \ldots, [x, x^n]\} = \mathrm{span}\{x^{m+1}, \ldots, x^n\}$. This means that $L^j = \mathrm{span}\{x^j, \ldots, x^n\}$ for $1 \leq j \leq n$ and $L^{n+1} = \{0\}$. In other words, L is nilpotent of class n.

Next assume that $x^{n+1} \neq 0$. In particular, assume $c_j = 0$ for all $j < k$ and $c_k \neq 0$. Let $1 \leq m \leq k$ and consider $[L, \mathrm{span}\{x^m, \ldots, x^n\}]$. Again, only left multiplication by x yields a non-zero result so that $[L, \mathrm{span}\{x^m, \ldots, x^n\}] = \mathrm{span}\{x^{m+1}, \ldots, x^n, x^{n+1}\}$. If $m < k$, $x^{n+1} = \sum_{\ell=k}^{n} c_\ell x^\ell \in \mathrm{span}\{x^{m+1}, \ldots, x^n\}$ so that $[L, \mathrm{span}\{x^m, \ldots, x^n\}] = \mathrm{span}\{x^{m+1}, \ldots, x^n\}$. If $m = k$ we have $x^{n+1} = c_k x^k + \sum_{\ell=k+1}^{n} c_\ell x^\ell$ with $c_k \neq 0$. Thus $\mathrm{span}\{x^{m+1}, \ldots, x^{n+1}\} = \mathrm{span}\{x^{k+1}, \ldots, x^{n+1}\} = \mathrm{span}\{x^k, \ldots, x^n\}$ and in this case $[L, \mathrm{span}\{x^k, \ldots, x^n\}] = \mathrm{span}\{x^k, \ldots, x^n\}$. In particular, $[L, \mathrm{span}\{x^m, \ldots, x^n\}] = \mathrm{span}\{x^{\min(k,m+1)}, \ldots, x^n\}$. This means $L^m = \mathrm{span}\{x^m, \ldots, x^n\}$ for $1 \leq m < k$ and $L^k = L^{k+1} = \cdots = \mathrm{span}\{x^k, \ldots, x^n\}$. Proposition 3.3 summarizes our findings.

Proposition 3.3. *Let L be an n-dimensional cyclic Leibniz algebra. Then either L is nilpotent of class n or $L \supsetneq L^2 \supsetneq \cdots \supsetneq L^k = L^{k+1} = \cdots \neq \{0\}$ for some $2 \leq k \leq n$. In this case, we say that L is* cyclic of type k. *Moreover, let x be* any generator *for L. Then L is nilpotent if and only if $x^{n+1} = 0$. If L is not nilpotent and of type k, then $x^{n+1} = \sum\limits_{\ell=k}^{n} c_\ell x^\ell$ for some $c_k, \ldots, c_n \in \mathbb{F}$ and $c_k \neq 0$. In particular, nilpotency and type do not depend on the choice of generator.*

As we turn our attention towards a classification of cyclic Leibniz algebras, again let $L \neq \{0\}$ be an n-dimensional cyclic Leibniz algebra generated by x with basis $\beta = \{x, x^2, \ldots, x^n\}$ and $x^{n+1} = \sum\limits_{j=2}^{n} c_j x^j$. Using an approach introduced in [BCGHHZ], we consider the left multiplication operator $\mathcal{L}_x : L \to L$ defined by $\mathcal{L}_x(z) = [x, z]$. We have that $\mathcal{L}_x(x^j) = x^{j+1}$ for $1 \leq j < n$ and

$\mathcal{L}_x(x^n) = \sum_{j=2}^{n} c_j x^j$. Thus we get the following coordinate matrix relative to the basis β:

$$[\mathcal{L}_x]_\beta = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & \cdots & 0 & 0 & c_2 \\ \vdots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & 0 & \vdots \\ 0 & & & \ddots & 1 & 0 & c_{n-1} \\ 0 & 0 & \cdots & \cdots & 0 & 1 & c_n \end{bmatrix}$$

The matrix $[\mathcal{L}_x]_\beta$ is the companion matrix to the polynomial $p(t) = t^n - c_n t^{n-1} - \cdots - c_2 t$ and thus the linear operator $\mathcal{L}_x$ has characteristic polynomial $p(t)$. Note that the polynomial $p(t)$ is in direct correspondence with our defining relation for x^{n+1}.

Suppose that $y = \sum_{i=1}^{n} b_i x^i \in L$. Then $\mathcal{L}_y(x^j) = \left[\sum_{i=1}^{n} b_i x^i, x^j\right] = \sum_{i=1}^{n} b_i [x^i, x^j] = b_1[x, x^j] = b_1 x^{j+1}$ since $[x^i, x^j] = 0$ for $i \geq 2$. This means $[\mathcal{L}_y]_\beta = b_1[\mathcal{L}_x]_\beta$. With only small, obvious modifications, the standard approach to determining the characteristic polynomial for a companion matrix (see, for example, [HC] Theorem 1 page 228) shows that the matrix $[\mathcal{L}_y]_\beta$ and thus the linear operator $\mathcal{L}_y$ has characteristic polynomial $t^n - b_1 c_n t^{n-1} - b_1^2 c_{n-1} t^{n-2} - \cdots - b_1^{n-1} c_2 t$. Note that if y is a generator[2] for L, using the correspondence between the characteristic polynomial of $\mathcal{L}_y$ and our defining relation for y^{n+1}, we see $y^{n+1} = \sum_{i=2}^{n} b_1^{n-i} c_i y^i$.

In summary for $n \geq 2$ and any $(c_2, \ldots, c_n) \in \mathbb{F}^{n-1}$ there is an n-dimensional cyclic Leibniz algebra L with generator x such that $\{x, x^2, \ldots, x^n\}$ is a basis for L and $x^{n+1} = \sum_{j=2}^{n} c_j x^j$. If y is any other generator with $y = \sum_{i=1}^{n} b_i x^i$ then $\{y, y^2, \ldots, y^n\}$ is a basis for L and $y^{n+1} = \sum_{j=2}^{n} b_1^{n-j} c_j y^j$. For $n \geq 2$, define an equivalence relation on $\mathbb{F}^{n-1}$ such that $(c_2, \ldots, c_n) \sim (b^{n-1}c_2, b^{n-2}c_3, \ldots, bc_n)$ for any $b \in \mathbb{F} - \{0\}$. Denote the equivalence classes as $[(c_2, \ldots, c_n)]$. This equivalence relation allows a simple classification of cyclic Leibniz algebras.

Theorem 3.4. *Up to isomorphism the only cyclic Leibniz algebras of dimensions* 0 *and* 1 *are the trivial* $\{0\}$ *algebra and the* 1*-dimensional abelian Lie algebra. For* $n \geq 2$*, up to isomorphism there is exactly one* n*-dimensional cyclic Leibniz algebra associated with each equivalence class* $[(c_2, \ldots, c_n)]$ *where* $(c_2, \ldots, c_n) \in \mathbb{F}^{n-1}$.

The nilpotent cyclic Leibniz algebras are associated with the class $[(0, \ldots, 0)] = \{(0, \ldots, 0)\}$. Cyclic Leibniz algebras of type k are associated with the class $[(0, \ldots, 0, c_k, \ldots, c_n)]$ for some $c_k, \ldots, c_n \in \mathbb{F}$ with $c_k \neq 0$. In this case, $\dim(L^k) = n - k + 1$ and $L^k = L^{k+1} = \cdots$.

The classification of complex cyclic Leibniz algebras obtained by Scofield and Sullivan [SS] split isomorphism classes of cyclic Leibniz algebras into cases of nilpotent or type k. For algebras of type k, they insist on a normalized generator such that $c_k = 1$. Note that their equivalence

[2] If y is a generator, we must have $b_1 \neq 0$. Otherwise, the algebra generated by y would be contained in $\text{span}\{x^2, \ldots, x^n\} \neq L$. Note also that for any $b_1 \neq 0$, $y = b_1 x$ is a generator.

class $[(c_{k+1}, \cdots, c_n)]$ corresponds to our class $[(0, \ldots, 0, 1, c_{k+1}, \ldots, c_n)]$. By avoiding this normalization we no longer need the existence of roots of unity and our equivalence relation is much simpler.

As in our construction, Batten Ray et. all. [BCGHHZ] identify the matrix for the left multiplication operator as a companion matrix to the polynomial $p(t)$. They use this observation as a tool to develop several important properties of cyclic Leibniz algebras. In particular they give a construction of the unique Cartan subalgebra for each cyclic Leibniz algebra, L, and in the process describe all maximal subalgebras of L as well as the minimal ideals of L and the unique maximal ideal of L.

4 A Class of Non-Lie, Non-Cyclic Leibniz Algebras

In this section we introduce a class of non-cyclic Leibniz algebras and study their properties. Fix some $n \geq 1$ and let L be the $n+1$ dimensional vector space with basis $\beta = \{x, x^2, \ldots, x^n, y\}$. To determine a bilinear operation on L it is enough to specify how multiplication works on basis elements.

Example 4.1. *Let L be the algebra with basis $\beta = \{x, x^2, \ldots, x^n, y\}$ and the bilinear bracket defined on the basis elements as follows: (1) $[x, x^j] = x^{j+1}$, $1 \leq j < n$; (2) $[x, x^n] = x^{n+1} = 0$; (3) $[x^k, x^j] = [x^k, y] = 0$ for all $2 \leq k \leq n$ and $1 \leq j \leq n$; (4) $[x, y] = x$, $[y, x^j] = -jx^j$ for $1 \leq j \leq n$; (5) $[y, y] = 0$.*

To see that L is a Leibniz algebra, we need to verify that the Leibniz identity holds. First, notice that $\langle x \rangle = \operatorname{span}\{x, x^2, \ldots, x^n\}$ forms a n-dimensional cyclic, nilpotent Leibniz subalgebra. Likewise, $\langle y \rangle = \operatorname{span}\{y\}$ forms a 1-dimensional cyclic Leibniz subalgebra which is an abelian Lie algebra. Thus we only need to check the Leibniz identity among triples of basis elements which involve both x and y.

First, triples that involve two y occurrences:

- *For $1 \leq j \leq n$, $[y, [y, x^j]] = 0 + [y, [y, x^j]] = [0, x^j] + [y, [y, x^j]] = [[y, y], x^j] + [y, [y, x^j]]$.*
- *For $2 \leq j \leq n$, $[x^j, [y, y]] = [x^j, 0] = 0 = 0 + 0 = [0, y] + [y, 0] = [[x^j, y], y] + [y, [x^j, y]]$ and for $j = 1$, $[x, [y, y]] = [x, 0] = 0 = x - x = [x, y] + [y, x] = [[x, y], y] + [y, [x, y]]$.*
- *For $2 \leq j \leq n$, $[y, [x^j, y]] = [y, 0] = 0 = 0 + 0 = [0, y] + [x^j, 0] = [[y, x^j], y] + [x^j, [y, y]]$ and for $j = 1$, $[y, [x, y]] = [y, x] = -x = -[x, y] + 0 = [-x, y] + [x, 0] = [[y, x], y] + [x, [y, y]]$.*

Finally, triples that involve one y occurrence:

- *Note that $[y, x^j] = -jx^j$ holds even when $j = n + 1$ since $x^{n+1} = 0$. Let $1 \leq k \leq n$.*
- *For $2 \leq j \leq n$, $[y, [x^j, x^k]] = [y, 0] = 0 = -j[x^j, x^k] = [-jx^j, x^k] + 0 = [[y, x^j], x^k] + [x^j, [y, x^k]]$ and for $j = 1$, $[y, [x, x^k]] = [y, x^{k+1}] = -(k + 1)x^{k+1} = [-x, x^k] + [x, -kx^k] = [[y, x], x^k] + [x, [y, x^k]]$.*
- *For $2 \leq j \leq n$, $[x^j, [y, x^k]] = 0 = 0 + 0 = [0, x^k] + [y, 0] = [[x^j, y], x^k] + [y, [x^j, x^k]]$ and for $j = 1$, $[x, [y, x^k]] = [x, -kx^k] = -kx^{k+1} = x^{k+1} - (k + 1)x^{k+1} = [x, x^k] + [y, x^{k+1}] = [[x, y], x^k] + [y, [x, x^k]]$.*

- *For* $2 \le j \le n$, $[x^j,[x^k,y]] = 0 = 0 + [x^k,0] = [[x^j,x^k],y] + [x^k,[x^j,y]]$ *and for* $j = 1$ *and* $k \ge 2$, $[x,[x^k,y]] = [x,0] = 0 = [x^{k+1},y] + 0 = [[x,x^k],y] + [x^k,[x,y]]$. *When* $j = k = 1$, $[x,[x,y]] = 0 + [x,[x,y]] = [[x,x],y] + [x,[x,y]]$.

We use the remainder of this section to investigate the structure of the Leibniz algebra L described in Example 4.1. Let us begin by determining the lower central series of L, $\mathrm{Leib}\,(L)$, and the derived series for L. Since none of the brackets output a y, $[L,L]$ must be contained in $\langle x\rangle = \mathrm{span}\{x,x^2,\ldots,x^n\}$. We have seen that $[-y,x] = x \in [L,L]$ and therefore $\langle x\rangle \subseteq [L,L]$ and hence $L^2 = [L,L] = \langle x\rangle$ In fact, it follows by induction that $L^k = \langle x\rangle$ for $k \ge 2$. We then have the lower central series

$$L = \mathrm{span}\{x,x^2,\ldots,x^n,y\} \supsetneq L^2 = L^3 = \cdots = \mathrm{span}\{x,x^2,\ldots,x^n\} \ne \{0\},$$

and thus L is not nilpotent.

Next observe $B = \mathrm{span}\left\{x^j \mid j \ge 2\right\}$ is an abelian ideal of codimension 2 in L so that $B \subseteq \mathrm{Leib}(L)$. Also, L/B is a Lie algebra and thus $\mathrm{Leib}(L) \subseteq B$. Therefore $\mathrm{Leib}(L) = B = \mathrm{span}\left\{x^j \mid j \ge 2\right\}$. Furthermore, since $[x+\mathrm{Leib}\,(L)\,, y+\mathrm{Leib}\,(L)] = [x,y]+\mathrm{Leib}\,(L) = x+\mathrm{Leib}\,(L)$, we have that $L/\mathrm{Leib}\,(L)$ is the non-abelian 2-dimensional Lie algebra. In addition, the derived series is given by

$$L^{(0)} = L \supsetneq L^{(1)} = \langle x\rangle \supsetneq L^{(2)} = \mathrm{Leib}(L) = \mathrm{span}\left\{x^j \mid j \ge 2\right\} \supsetneq L^{(3)} = \{0\}$$

and thus L is solvable.

Could it be that L is simply a sum of cyclic Leibniz algebras? Recall that for a cyclic Leibniz algebra C, $C/\mathrm{Leib}\,(C)$ is the 1-dimensional abelian Lie algebra. Thus if $M = C_1 \oplus \cdots \oplus C_\ell$ is a Leibniz algebra direct sum of cyclic Leibniz algebra $C_1,\ldots,C_\ell$, then $M/\mathrm{Leib}\,(M) = (C_1 \oplus \cdots \oplus C_\ell)/(\mathrm{Leib}\,(C_1) \oplus \cdots \oplus \mathrm{Leib}\,(C_\ell)) \cong (C_1/\mathrm{Leib}\,(C_1)) \oplus \cdots \oplus (C_\ell/\mathrm{Leib}\,(C_\ell))$ and so $M/\mathrm{Leib}\,(M)$ is a direct sum of 1-dimensional abelian Lie algebras. In other words, $M/\mathrm{Leib}\,(M)$ is the ℓ-dimensional abelian Lie algebra. Since $L/\mathrm{Leib}\,(L)$ is not abelian, L is neither cyclic nor a (Leibniz algebra) direct sum of cyclic subalgebras.

Also, since L is solvable, $L = \mathrm{rad}(L)$ and so L is (unsurprisingly) not semisimple. Also, $\mathrm{span}\{x^m,x^{m+1},\ldots,x^n\}$ for $1 \le m \le n$ are easily seen to be ideals. In particular, $\mathrm{span}\{x,x^2,\ldots,x^n\}$ is an ideal distinct from $\{0\}$, $\mathrm{Leib}\,(L)$, and L so that L is not simple. In summary,

Theorem 4.2. *The Leibniz algebra* $L = \mathrm{span}\{x,x^2,\ldots,x^n,y\}$ *with bracket structure given in Example 4.1 is not nilpotent, semisimple, or simple. But L is solvable. Its maximal Lie algebra homomorphic image,* $L/\mathrm{Leib}\,(L)$, *is the non-abelian* 2*-dimensional Lie algebra. Consequently L is not a (Leibniz algebra) direct sum of cyclic Leibniz algebras.*

5 Adjoining a Module

In this section we offer a second class of examples. By first extending the familiar Lie algebra construction of adjoining a module to an algebra to the context of Leibniz algebras and then considering adjoining a cyclic module to a nilpotent cyclic Leibniz algebra, we obtain a class of algebras with similar properties to those of the previous section except here we will have that the maximal Lie algebra homomorphic image is abelian.

Definition 5.1. *Let L be a Leibniz algebra and M a vector space over $\mathbb{F}$ equipped with bilinear maps $[\ ,\ ] : L \times M \to M$ and $[\ ,\ ] : M \times L \to M$ (a left and a right action) such that for all $a, b \in L$ and $m \in M$ the following hold:*

1. $[a, [b, m]] = [[a, b], m] + [b, [a, m]]$
2. $[a, [m, b]] = [[a, m], b] + [m, [a, b]]$
3. $[m, [a, b]] = [[m, a], b] + [a, [m, b]]$

We note that if L is a Lie algebra with L-module M and action $x{\cdot}m$ for $x \in L$ and $m \in M$, then left action $[x, m] = x{\cdot}m$ and right action $[m, x] = -x{\cdot}m$ turn M into a module viewing L as merely a Leibniz algebra.

Example 5.2. *Let $L = \operatorname{span}\{x, x^2, \ldots, x^n\}$ be the n-dimensional nilpotent cyclic Leibniz algebra. Consider the vector space $M = \operatorname{span}(\beta)$ with basis $\beta = \{y_1, y_2, \ldots, y_n\}$. Let $2 \le j \le n$ and $1 \le k \le n$ and define $[x^j, y_k] = 0$. When $k < n$ define $[x, y_k] = y_{k+1}$ and let $[x, y_n] = 0$. For convenience let $y_{n+1} = 0$ so that $[x, y_k] = y_{k+1}$ for all $1 \le k \le n$. Finally, let $[y_k, x^j] = 0$ for all $1 \le j \le n$ and $1 \le k \le n$. In other words, the right action of L on M is trivial whereas x acts in cyclic fashion on the left.*

With these definitions, M is an L-Module. To see this we must verify the relations in Definition 5.1. In relation 1, all terms are zero unless $a = b = x$. In this case relation 1 becomes $[x, [x, m]] = [[x, x], m] + [x, [x, m]]$ which is clearly true since $[[x, x], m] = [x^2, m] = 0$. Relations 2 and 3 hold because all terms are zero as they each involve the trivial right action of L.

We show in the following proposition that for L a Leibniz algebra and M an L-module, the vector space direct sum $L \oplus M$ becomes a Leibniz algebra if for $x_1, x_2 \in L$ and $m_1, m_2 \in M$ we define $[x_1 + m_1, x_2 + m_2] = [x_1, x_2] + [x_1, m_2] + [m_1, x_2]$. Notice that in the definition of the bracket on $L \oplus M$, $[x_1, x_2]$ is the bracket in L, $[x_1, m_2]$ is the left action of L on M, and $[m_1, x_2]$ is the right action of L on M.

Proposition 5.3. *Let L be a Leibniz algebra and M an L-module. The vector space direct sum $L \oplus M$ becomes a Leibniz algebra if for $x_1, x_2 \in L$ and $m_1, m_2 \in M$ we define $[x_1 + m_1, x_2 + m_2] = [x_1, x_2] + [x_1, m_2] + [m_1, x_2]$. Moreover, L is a subalgebra and M is an abelian ideal of $L \oplus M$.*

Proof: It is obvious that the bracket on $L \oplus M$ is bilinear. We need to verify the Leibniz identity. Let $x_1, x_2, x_3 \in L$ and $m_1, m_2, m_3 \in M$. Consider the following brackets:

$$\begin{aligned}
\underbrace{[x_1 + m_1, [x_2 + m_2, x_3 + m_3]]}_{\text{LM}_\text{A}} &= [x_1 + m_1, [x_2, x_3] + [x_2, m_3] + [m_2, x_3]] \\
&= \underbrace{[x_1, [x_2, x_3]]}_{\text{Leibniz}_\text{A}} + \underbrace{[x_1, [x_2, m_3]]}_{1_\text{A}} + \underbrace{[x_1, [m_2, x_3]]}_{2_\text{A}} + \underbrace{[m_1, [x_2, x_3]]}_{3_\text{A}}
\end{aligned}$$

$$\begin{aligned}
\underbrace{[[x_1 + m_1, x_2 + m_2], x_3 + m_3]}_{\text{LM}_\text{B}} &= [[x_1, x_2], x_3 + m_3] + [[x_1, m_2], x_3 + m_3] + [[m_1, x_2], x_3 + m_3] \\
&= \underbrace{[[x_1, x_2], x_3]}_{\text{Leibniz}_\text{B}} + \underbrace{[[x_1, x_2], m_3]}_{1_\text{B}} + \underbrace{[[x_1, m_2], x_3]}_{2_\text{B}} + \underbrace{[[m_1, x_2], x_3]}_{3_\text{B}}
\end{aligned}$$

$$\underbrace{[x_2+m_2,[x_1+m_1,x_3+m_3]]}_{\text{LM}_\text{C}} = [x_2+m_2,[x_1,x_3]]+[x_2+m_2,[x_1,m_3]]+[x_2+m_2,[m_1,x_3]]$$

$$= \underbrace{[x_2,[x_1,x_3]]}_{\text{Leibniz}_\text{C}}+\underbrace{[m_2,[x_1,x_3]]}_{2_\text{C}}+\underbrace{[x_2,[x_1,m_3]]}_{1_\text{C}}+\underbrace{[x_2,[m_1,x_3]]}_{3_\text{C}}$$

The module axioms 1, 2, and 3 for M guarantee that $1_\text{A} = 1_\text{B} + 1_\text{C}$, $2_\text{A} = 2_\text{B} + 2_\text{C}$, and $3_\text{A} = 3_\text{B} + 3_\text{C}$. The Leibniz identity for L guarantees that $\text{Leibniz}_\text{A} = \text{Leibniz}_\text{B} + \text{Leibniz}_\text{C}$. Putting these together we see that $\text{LM}_\text{A} = \text{LM}_\text{B} + \text{LM}_\text{C}$ and so the Leibniz identity holds on $L \oplus M$. $\square$

Taking L and M as defined in Example 5.2, let $K = L \oplus M = \text{span}\{x, x^2, \ldots, x^n, y_1, \ldots, y_n\}$. We have that K is a Leibniz algebra using the above construction and can now investigate the structure of this algebra.

For $x \in L$ and $m \in M$, we have $[x+m, x+m] = [x,x] + [x,m] + [m,x]$. Therefore, $\text{Leib}\,(L \oplus M) = \text{Leib}\,(L) \oplus \text{span}\{[x,m] + [m,x] \mid x \in L \text{ and } m \in M\}$, where $\oplus$ represents a vector space direct sum. Furthermore, we know that $\text{Leib}\,(L) = \text{span}\{x^2, \ldots, x^n\}$ and all brackets (i.e., actions) between L and M either output 0 or something in $\text{span}\{y_2, \ldots, y_n\}$. In fact, $[x, y_k] + [y_k, x] = y_{k+1} + 0 = y_{k+1} \in \text{span}\{[x,m]+[m,x] \mid x \in L \text{ and } m \in M\}$ for $1 \le k \le n$. Therefore, $\text{Leib}\,(K) = \text{span}\{x^2, \ldots, x^n, y_2, \ldots, y_n\}$.

Next we explicitly calculate the lower central series for K. First, looking at the brackets for K we see that they never output any power of x smaller than x^2 and never output y_1. Thus $[K,K] \subseteq \text{span}\{x^2, \ldots, x^n, y_2, \ldots, y_n\}$. But by definition, $\text{Leib}\,(K) \subseteq [K,K]$. Therefore, $[K,K] = \text{Leib}\,(K) = \text{span}\{x^2, \ldots, x^n, y_2, \ldots, y_n\}$. We claim that $K^\ell = \text{span}\{x^\ell, \ldots, x^n, y_\ell, \ldots, y_n\}$ for $1 \le \ell \le n$ and $\{0\} = K^{n+1} = K^{n+2} = \ldots$ so that K is nilpotent of class n. We proceed by induction, notice that $[x, K^\ell] = \text{span}\{x^{\ell+1}, \ldots, x^{n+1}, y_{\ell+1}, \ldots, y_{n+1}\}$ where for convenience we let $x^m = y_m = 0$ for $m > n$. Also, $[x^j, K^\ell] = [y, K^\ell] = \{0\}$ for $j \ge 2$. The result follows and from it we observe that L is nilpotent.

Note that we could forgo the explicit construction of the lower central series and still arrive at the nilpotency of K by applying a theorem of Bosko et. all. [BHHSS]. Every left multiplication by an element of L on K is nilpotent and trivially left multiplication on K by elements from M are nilpotent. Therefore since $L \cup M$ is a Lie set (i.e., it is closed under brackets and spans K), Jacobson's refinement of Engel's theorem for Leibniz algebras [BHHSS] shows $K = L \oplus M$ is nilpotent.

Next we examine the structure of the cyclic subalgebras of K. Let $z = \sum_{i=1}^{n} a_i x^i + \sum_{j=1}^{n} b_j y_j \in K$.

Then

$$
\begin{aligned}
z^2 &= [z,z] = a_1 \sum_{i=1}^{n-1} a_i x^{i+1} + a_1 \sum_{j=1}^{n-1} b_j y_{j+1} = \sum_{i=2}^{n} a_1 a_{i-1} x^i + \sum_{j=2}^{n} a_1 b_{j-1} y_j \text{ and} \\
z^3 &= [z,z^2] = a_1 \sum_{i=2}^{n-1} a_1 a_{i-1} x^{i+1} + a_1 \sum_{j=2}^{n-1} a_1 b_{j-1} y_{j+1} = \sum_{i=3}^{n} a_1^2 a_{i-2} x^i + \sum_{j=3}^{n} a_1^2 b_{j-2} y_j.
\end{aligned}
$$

In general,

$$
z^\ell = \sum_{i=\ell}^{n} a_1^{\ell-1} a_{i-\ell+1} x^i + \sum_{j=\ell}^{n} a_1^{\ell-1} b_{j-\ell+1} y_j \text{ for } 1 \leq \ell \leq n \text{ and } z^\ell = 0 \text{ for } \ell > n.
$$

As a consequence, if $a_1 = 0$, then $z^2 = 0$. If $a_1 \neq 0$ and $1 \leq \ell \leq n$ then the coefficient of x^ℓ in z^ℓ is $a_1^{\ell-1} a_{\ell-\ell+1} = a_1^\ell \neq 0$. In all cases $z^{n+1} = 0$ and thus by proposition 3.3 all cyclic subalgebras, $\langle z \rangle$, are nilpotent. For $n > 1$ they are either trivial ($z = 0$), 1-dimensional ($z \neq 0$ but $a_1 = 0$), or n-dimensional ($a_1 \neq 0$). For $n = 1$, they are either trivial or 1-dimensional. Our understanding of the cyclic subalgebras of K plays a key role in understanding the structure of this Leibniz algebra.

Theorem 5.4. *The Leibniz algebra* $K = \operatorname{span}\{x, x^2, \ldots, x^n, y_1, y_2, \ldots, y_n\}$ *with brackets given in Example 5.2 and Proposition 5.3 is neither semisimple nor simple. But* K *is nilpotent of class* n *and solvable. Its maximal Lie algebra homomorphic image,* $K/\operatorname{Leib}(K)$*, is the* 2*-dimensional abelian Lie algebra. Also, for* $n > 1$*,* K *is not a (Leibniz algebra) direct sum of cyclic Leibniz algebras.*

Proof: We have already seen that K is nilpotent. Since K is nilpotent, it is also solvable. Referring back to definitions, it is obvious that K is neither simple nor semisimple. By definition, $K/\operatorname{Leib}(K) = \operatorname{span}\{x + \operatorname{Leib}(K), y_1 + \operatorname{Leib}(K)\}$. Notice that $[x + \operatorname{Leib}(K), y_1 + \operatorname{Leib}(K)] = [x, y_1] + \operatorname{Leib}(K) = y_2 + \operatorname{Leib}(K) = 0 + \operatorname{Leib}(K)$, since $y_2 \in \operatorname{Leib}(K)$. Hence $K/\operatorname{Leib}(K)$ is the 2-dimensional abelian Lie algebra.

Suppose that K is a (Leibniz algebra) direct sum of cyclic Leibniz algebras. We have seen previously that if $C = C_1 \oplus \cdots \oplus C_\ell$ is a direct sum of cyclic algebras then $C/\operatorname{Leib}(C) = C_1/\operatorname{Leib}(C_1) \oplus \cdots \oplus C_\ell/\operatorname{Leib}(C_\ell)$ and that each $C_i/\operatorname{Leib}(C_i)$ is the one-dimensional abelian algebra. Thus if K is a (Leibniz algebra) direct sum of cyclic subalgebras, it must be a sum of exactly $\dim(K/\operatorname{Leib}(K)) = 2$ subalgebras. Considering that cyclic subalgebras of K have dimensions $0, 1$, and n and that $\dim(K) = 2n$, we must have two cyclic subalgebras of dimension n. Suppose that $K = \langle z_1 \rangle \oplus \langle z_2 \rangle$ where $z_1 = \sum_{i=1}^{n} a_i x^i + \sum_{j=1}^{n} b_j y_j$ and $z_2 = \sum_{i=1}^{n} c_i x^i + \sum_{j=1}^{n} d_j y_j$. Since these are n-dimensional subalgebras we must have $a_1 \neq 0$ and $c_1 \neq 0$. But then $[z_1, z_2] = a_1 \sum_{i=1}^{n-1} c_i x^{i+1} + a_1 \sum_{j=1}^{n-1} d_j y_{j+1}$. Notice that the coefficient of x^2 in $[z_1, z_2]$ is $a_1 c_1 \neq 0$. Since $[z_1, z_2] \neq 0$, this is not a Leibniz algebra direct sum (contradiction). $\square$

Note that when $n = 1$, $K = \operatorname{span}\{x, y_1\}$ where $[x, x] = [x, y_1] = [y_1, x] = [y_1, y_1] = 0$ so K is the 2-dimensional abelian Lie algebra and is in this trivial situation a direct sum of cyclic subalgebras. For example, one such decomposition is $K = \langle x \rangle \oplus \langle y_1 \rangle$.

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References

[BCGHHZ] Batten Ray, C. A. Combs, N. Gin, A. Hedges, J.T. Hird, L. Zack. Nilpotent Lie and Leibniz algebras. Communications in Algebra, 24: 2404 – 2410, 2014.

[DMS] Demir I., K.C. Misra, and E. Stitzinger. On some structures of Leibniz algebras. Contemporary Mathematics, 623: 41–54, 2014.

[EW] Erdmann, K., M.J. Wildon. (2006). *Introduction to Lie algebras*, Springer Undergraduate Mathematics Series, Springer-Verlag London Ltd., London.

[HC] Hoffman, K. , R. Kunze. (1971). *Linear Algebra*, 2nd edition, Prentice-Hall, Englewood Cliffs, New Jersey.

[BHHSS] Bosko, L., A. Hedges, J.T. Hird, N. Schwartz, K. Stagg. Jacobson's refinement of Engel's theorem for Leibniz algebras, Involve, 4: 293 –296, 2011.

[L] Loday, J.-L., Une version non commutative des algebres de Lie: les algebres de Leibniz, Enseign. Math. (2), 1993, 39(3-4): 269–293.

[SS] Scofield, D., S. McKay Sullivan. Classification of Complex Cyclic Leibniz Algebras. https://arxiv.org/abs/1411.0170