Quantifying the Curvature of Curves: An Intuitive Introduction to Differential Geometry

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Abstract

In this paper we introduce the reader to a foundational topic of differential geometry: the curvature of a curve. To make this topic engaging to a wide audience of readers, we develop this intuitive introduction employing only basic geometry without calculus and derivatives. It is hoped that this introduction will encourage many more to both consider this mathematical notion and to develop enthusiasm for mathematical studies.

1 Introduction

Ride a bike or drive a car. Hopefully, well before you end up in a ditch, you will recognize that not all curves in a road are constructed equally: some curves are simply *sharper* or more *curved* than others. The subject of differential geometry leads to the measuring (or quantifying) of curve curvature. Unfortunately, formally investigating differential geometry at an introductory level requires at least differential and integral calculus and linear algebra. Further, differential equations, tensor calculus as well as manifold theory are necessary for more advanced treatments of the subject. However, an intuitive understanding of curve curvature is approachable by high school and college students and their instructors.

The main goal of this article is to widen the audience of, and appreciation for, differential geometry by developing the notion of curve curvature in an intuitive manner. We accomplish this goal in a number of ways: First, we introduce, explore, and develop the notion of curvature using elementary geometry and a minimal number of equations. Second, we have provided a number of online, dynamic applets throughout the paper through which the reader can investigate and interact with these ideas directly without needing to manipulate equations. Third, we also provide exploration questions, some of which make use one of the applets, that could be used as homework or classroom projects. Growing from the authors' experiences in teaching, these explorations naturally fit into courses such as high school geometry, general liberal arts math, and the calculus sequence. Last, the mathematical ideas introduced in the article are recreated and extended in the form of a graphic novel. Altogether, these approaches widen the readership and applicability of this exploration into curve curvature.

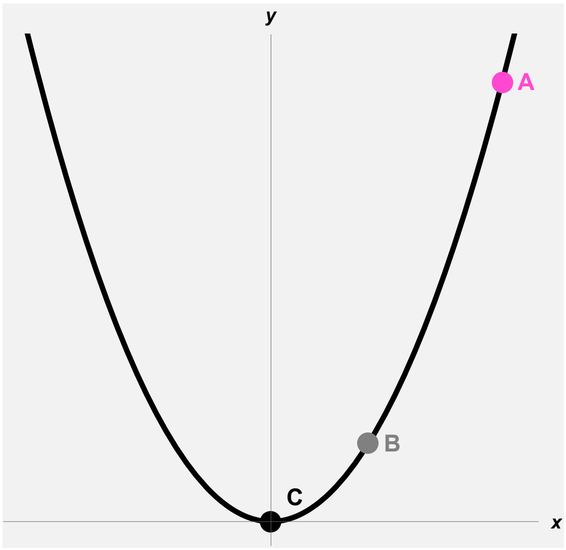


Figure 1: **Curvature Intuitively**. This parabola is most bent at C (the vertex) and least bent at A. How do we measure this?

Curvature is recognized as one of the most fundamental topics in differential geometry. In fact, M. Spivak devotes an entire volume [S.] to the study of curvature within a historical framework. This volume is a wonderful, modern treatment of the notion of curvature as it evolved from the original investigations of L. Euler to later extensions by C. F. Gauss and B. Riemann. In the text, [O.] differential geometry is approached from the viewpoint of E. Cartan. Aiming to present notions of differential geometry to a wider audience with less mathematical background, B. O'Neill states "*This book is an elementary account of geometry of curves and surfaces. It is written for students who have completed standard courses in calculus and linear algebra, and its aim is to introduce some of the main ideas of differential geometry*" ([O.] page ix). This paper seeks to further simplify the topic of curvature by making it accessible to students who possess a rudimentary understanding of only geometry and algebra. Considering only lines, circles, and ratios, we present an intuitive, geometric understanding of this subject. It is our hope that students may be interested in continuing on to a more rigorous treatment of the subject in the future.

Without reading advanced texts, everyone has an intuitive understanding of curvature; shapes like straight lines (___) have no curvature while shapes like the curve ($\sim$) are curved. The idea of curvature is even built into the names of these shapes (straight and curved). Delving deeper, one sees the challenge is not so much to say *"Yes, this shape is curved."* or *"No, this shape is not curved."*, but rather to ask, for example, *"How curved is the shape at one point versus another point?"* (see Figure 1). This more nuanced question will be addressed and answered in this article.

Modern students, digital natives, are inundated with interactive, dynamic, graphical information and are, therefore, more attuned to visual cues and information delivery than previous generations [P.]. As software and device usage (e.g., apps, tablets, smartphones) become ubiquitously integrated into all aspects of our lives, it seems fitting that students should be offered increased opportunities to dynamically interact with mathematical concepts. To meet the needs of these students this article employs three novel learning aids for the study of curve curvature: a limited pre-calculus–based tool set; a variety of interactive software demos; and a graphic novel style in select sections. The interactive demos can be downloaded from the web and run on a MAC or PC using the free Wolfram CDF player.[1] These dynamic visualizations will communicate the beauty of mathematics to a diverse audience in a manner far beyond that of static equations. The *graphically intensive* treatment is stylistically similar to a *cartoon* or *graphic novel*. This, too, is common among this generation's school-age readers. Through this modality, images and text reverse roles; visuals are highlighted with text playing a supporting role. These visual and software interactives will help connect the subject of differential geometry and curve curvature to a broader audience including high school students and teachers.

[1] https://mathsci2.appstate.edu/~osbornejm/InteractiveDemos.html

2 Tangent and Normal Vectors for Plane Curves

Imagine that you have a obtained a straight piece of thin, malleable, metallic wire. Lay this wire on a flat surface like your kitchen table and then begin to bend this wire into a curved shape of your choice. The only caveat is that, once you have finished bending, your final curve must be able to lay perfectly flat. A curve made in this fashion is called a *plane curve* (see Figure 2). Try to make sure that you don't introduce any kinks or crossings into your wire creation. That is, make sure that at no place does it have a kink like ⅄. Also avoid wire crossings like ↬ as they will add thickness to your curve and cause it to not lay flat. Now imagine that onto your metal creation you place a small magnetic ball (●) that can be rolled from one end to the other. Note that it will be possible to roll this small ball along the wire because there are no kinks or crossings on which to get stuck.

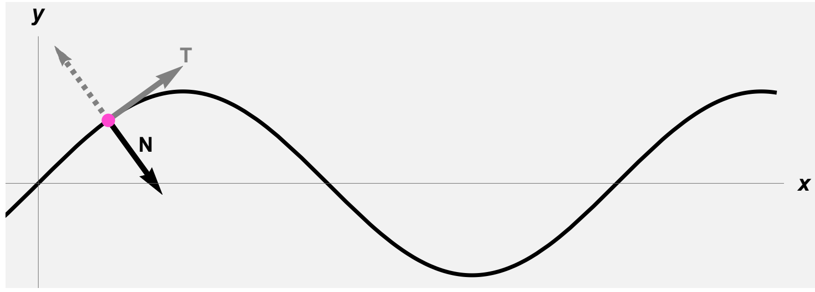


Figure 2: **A Tangent (T) and Normal (N) on a Sine Wave.** The normal is a vector perpendicular to the tangent, how do we choose which one?

As the magnetic ball moves along the plane curve, two vectors (i.e. arrows), the tangent (T) and normal (N) are created (see Figure 2). Imagine the ball loses its magnetic grip, the ball would fly away in the direction given by the vector T. In other words, the direction of motion vector will be tangent to the curve at the instance where the magnetism is lost. Note that, in our discussion, we have implicitly chosen a beginning and end of our curve (typically from left to right). If the choice of beginning and end is reversed, then the direction of every T will also be reversed. The vector N is perpendicular to T at each point on the curve. However, since two vectors are perpendicular to T (see Figure 2), to uniquely define N, a choice must be made. Here is how you choose: because of the bends built into the curve, the vector T changes direction as the magnetic ball rolls along the curve. That is, the *turning* of the curve changes T. The vector N is chosen based on its ability to predict future *turns* in the curve.

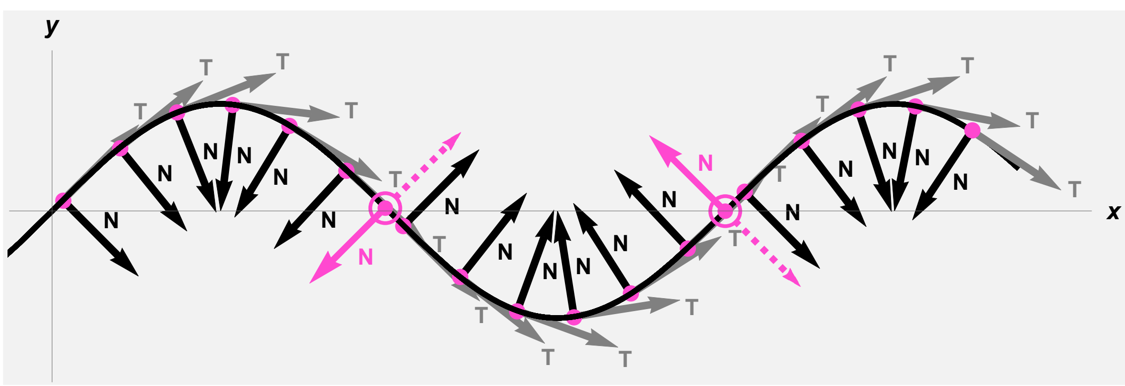


Figure 3: **The Normal Vector.** The vector N predicts the direction in which the vector T will turn. Note that at inflection points, denoted by ⊙, the normal vector abruptly changes direction.

Notice that, during the first bend in the curve in Figure 3, the vector T is turning *down* as the magnetic ball moves from left to right, as was predicted by the normal vector N pointing *down*. In the next bend, the vector T turns *up*, again predicted by the normal vector N pointing *up*. There are points on a curve where the normal vector is not uniquely defined. At inflection points, denoted by ⊙ in Figure 3, the vector T momentarily stops changing. At such a point, predicting future tangents is impossible, and so we arbitrarily choose one of the two possible N's. The vectors N will abruptly change direction from pointing *down/up* to *up/down* at these points of inflection.

3 Curves of Constant Curvature: Lines and Circles

Now that we know how T and N work together to indicate how a plane curve *turns*, we can say that:

At each point, *curve curvature* quantifies the degree of deviation of the curve from a straight line.

The change in T is used to precisely quantify this deviation. Intuitively, if a curve has a *sharp* turn and so T is changing *rapidly*, then the curvature should have a *large* value. In contrast, if a turn is *gradual*, and so T is changing slowly, then curvature should be *small*. Note that since T and N change together we can quantify these deviations using N just as well as T. When discussing *lines*, studying the change in T is natural, while in the case of *circles*, N is a better choice.

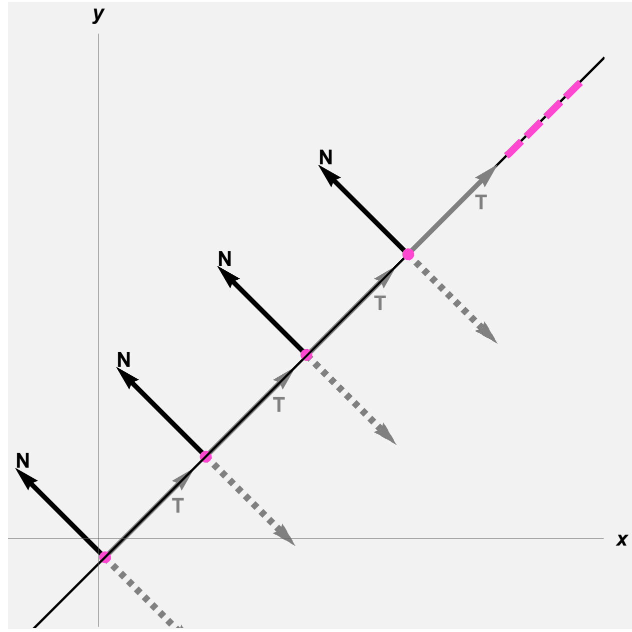


Figure 4: **A Line Has Zero Curvature.**

When a curve has a *constant* curvature value from one location to another, one can imagine the curve as being constructed from a series of uniformly bent components. The first example of a curve of constant curvature is a *line*. The line (———) can be viewed as an object comprised of a series of the smaller *straight–line–components* (– –) each with no curvature (see Figure 4). Notice also in Figure 4 the direction vector T does not turn. Since T does not change, the curvature should have a constant value 0. Recall that when T does not change, N is not uniquely determined. In Figure 4 we made a choice of N pointing up; but N pointing down is equally valid.

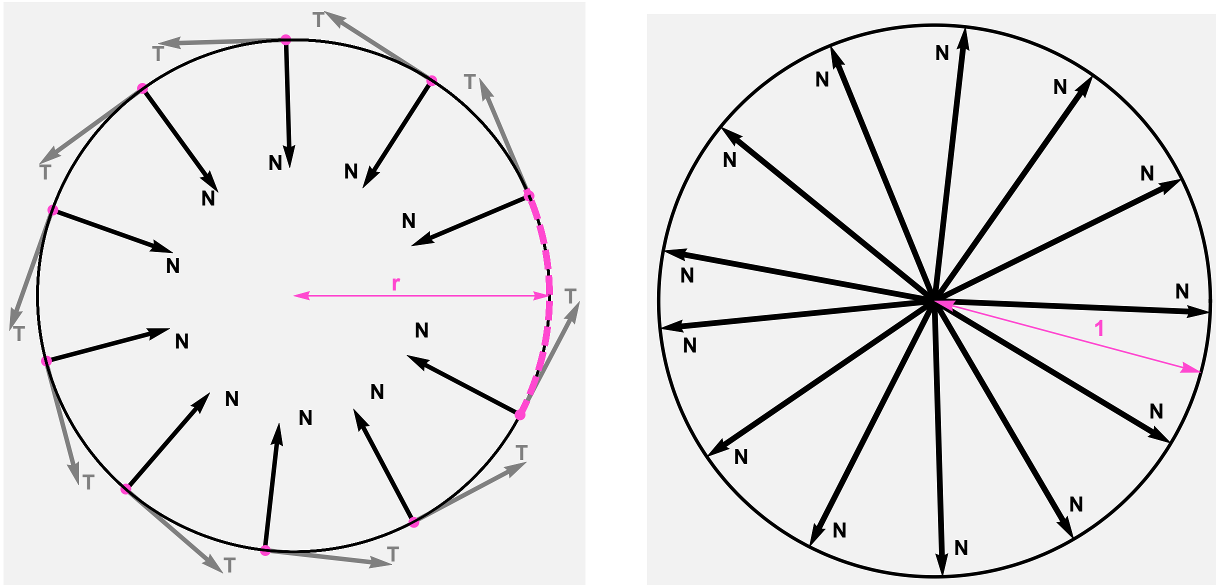
A *circle* is another example of a curve with constant curvature. Like a toy train set which has been shipped to you as an unassembled set of smaller track pieces, one may build a complete circular track from multiple copies of a single circular–arc of constant curvature. Since a circle deviates from a straight–line in the same way at every point, while its curvature should be constant, it should not be zero.

Let us now quantify the non-zero value describing this curvature. Recall that curvature can be determined by the change in the normal vector. Imagine a magnetic ball (●) rolling around a circle. As the ball makes one full transit around the circle it travels a distance of $2\pi r$. Figure 5(b) shows that as the ball make one full circuit, the normal vector itself makes one full turn. Therefore the tip-end of N travels a distance of 2π as the ball travels a distance of $2\pi r$. Curvature is the ratio of these two distances.

Therefore, the curvature, denoted $\kappa_\odot$, of a circle of radius r is given by the mathematical formula,

$$\kappa_\odot = \frac{\text{distance traveled by } N \text{ turning}}{\text{distance traveled by } (\bullet)} = \frac{2\pi}{2\pi r} = \frac{1}{r}. \quad (1)$$

We can now notice that the curvature of a circle is the reciprocal of its radius. Let us now discuss why this formula matches our intuitive understanding of curvature. Imagine the magnetic ball on a circular track with a large radius. From the perspective of the ball, the track will appear to have a gradual turn; the corresponding curvature should have a very small value. Notice that the reciprocal of a large radius gives us a small curvature. Alternatively, if the track has a small radius, then, from the perspective of the ball, the track will appear to have a sharp turn. The reciprocal of a small radius gives us a large curvature. So, while at first glance the curvature of a circle being the reciprocal of its radius might seem strange, in retrospect it perfectly matches our intuition.



(a) **The Vectors T and N on the Circle.** Because T and N change uniformly, a circle has a constant, non-zero curvature.

(b) **Normal Vectors N on the Unit Circle.** As the ball travels once around the circle pictured left, N turns once around the unit circle.

Figure 5: **A Circle of Radius r: A Curve of Constant, Non-Zero Curvature.**

4 The Osculating Circle of a Plane Curve and Curve Curvature

Equation (1) relates the curvature of a circle to its radius, and is the key to developing a tool that can be applied to curves of non-constant curvature. This tool is called the *osculating circle*. Except at points where the curve is *perfectly* flat, there is a circle of *some* radius that *osculates* (which means, *kisses*) the curve. The osculating circle can be viewed as the best circular approximation of the curve at that instant. This means that this circle is of the perfect radius so as to conform to the shape of the curve at that point. Figure 6 shows that as we zoom in the circle perfectly matches the curve at that point.

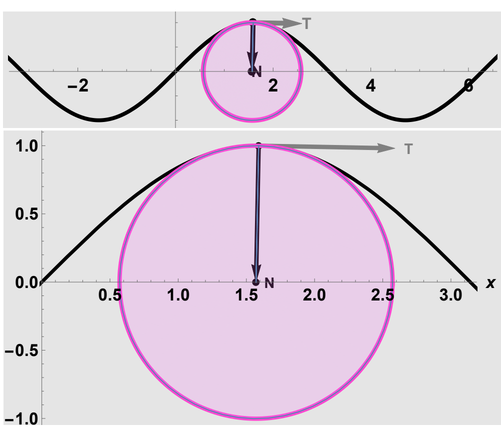


Figure 6: **An Osculating Circle.** The circle of radius 1 is the best circular approximation of this function at this location.

As shown in Figure 7, this circle's radius can change based on the location it *kisses*. That is, for the circle to conform to the contours of the curve it must be able to change its shape (and therefore, its radius) accordingly. In fact, Figure 7 shows a tacit relationship between the radius of the osculating circle and the curvature of the curve. At locations where the curve has a large curvature, for example at the point where the left–most circle kisses the curve, the radius of the osculating circle is small. The opposite is true at the point where the right most circle kisses the curve; the curve is flatter and so the radius of the osculating circle is larger. The middle circle kisses a point where the curve appears to be nearly flat (almost line-like), and therefore the radius of the osculating circle is much larger. These observations strongly suggest that we define the curvature of a curve at a given point as simply the reciprocal of the radius of the osculating circle at that point.

At a point where the curve is *perfectly* flat, we define the curvature to be 0. If we try to draw an osculating circle at such a point, we can never make the radius large enough for the circle to conform to the curve. For the circle to conform, it would have to have an *infinite* radius or, in other words, the best approximation at such a point is a line and not a circle. As its radius grows, a circle limits to a line. Likewise, the reciprocal of its radius limits to 0.

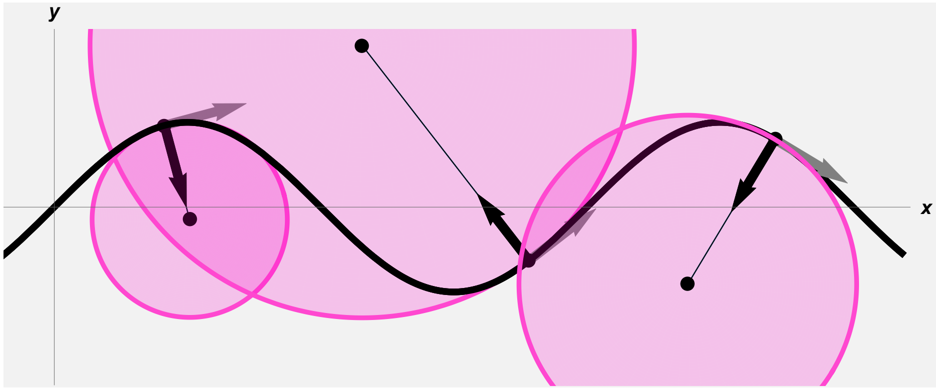


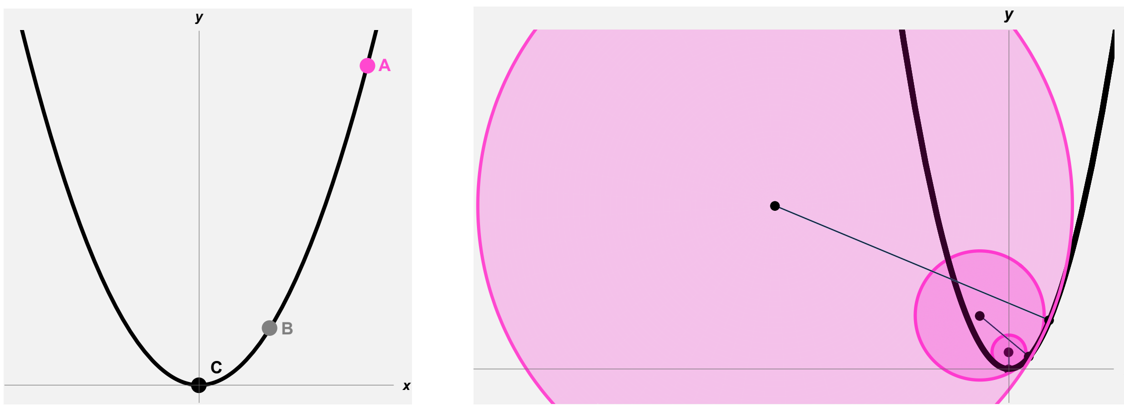
Figure 7: **A Sine Curve with Osculating Circles.** At each point, an osculating circle has the perfect radius so as to conform to the curve. Note the inverse relationship; big circles correspond with little curvature and vice-versa.

Think back to the curve you created by bending your long piece of straight wire. Your curve will most likely not have constant curvature since you will have decided to make a more interesting shape than a line or a circle. And while your curve, on the whole, is not a line or a circle, you can think of your curve as being constructed from a series of various sized straight–line segments and circular bends. At flat points the curvature is defined to be 0, while at any other point the curvature of the curve is defined to be the curvature of its osculating circle. That is, as a mathematical formula, we define the curvature $\kappa_\odot$ of generic curve as 0 at flat points and otherwise as $\kappa_\odot = 1/r_\odot$ where $r_\odot$ is the radius of the osculating circle.

5 Example: A Parabola Revisited

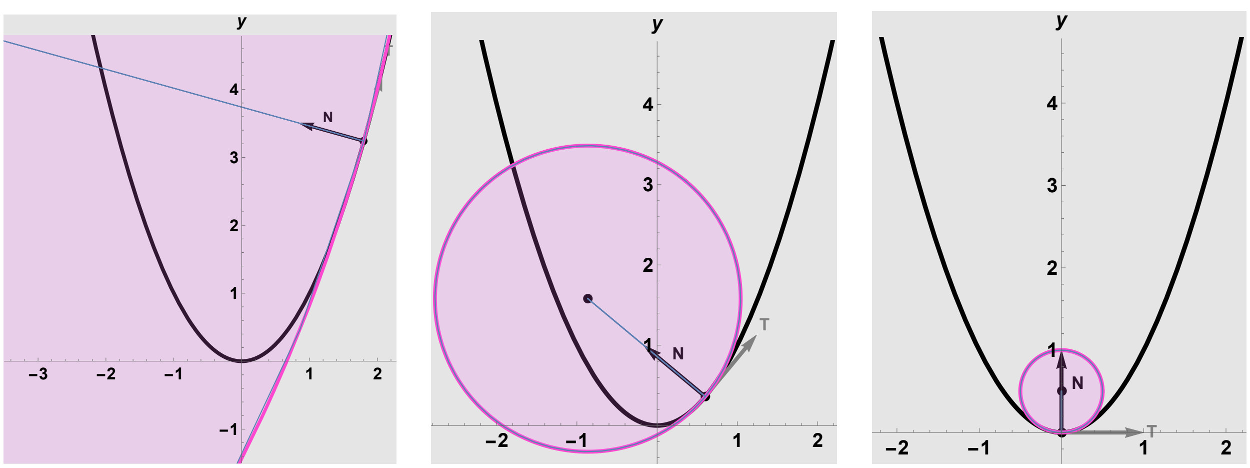
Equipped with an intuitive, yet formal definition of curve curvature, it is possible to answer some more nuanced differential geometric questions regarding curvature.

- **(Question)** Which points on the parabola in Figure 8(a) are the most and least curved?
- **(Answer)** The answer to this question is simply an application of drawing osculating circles, computing their reciprocals, and then ordering the resulting values. (See Figure 9.)



(a) **A Parabola.** (b) **A Parabola with Osculating Circles.**

Figure 8: **Curvature for a Parabola.** Considering the points A, B, and C, where does the parabola have *largest* and *smallest* curvatures? A careful answer requires analyzing the osculating circles at these points.



(a) (Point A, $r_{\odot} \approx 26.08$) (b) (Point B, $r_{\odot} \approx 1.91$) (c) (Point C, $r_{\odot} = 0.5$)

Figure 9: **Curvature Values at A, B, and C.** Computing the reciprocals of the radius of the osculating circles gives the answer as A (least curved, $\kappa_{\odot} \approx 0.04$), B (in the middle, $\kappa_{\odot} \approx 0.52$), then C (most curved, $\kappa_{\odot} = 2$) as one moves towards the vertex of the parabola.

6 Example: The Fresnel Spiral

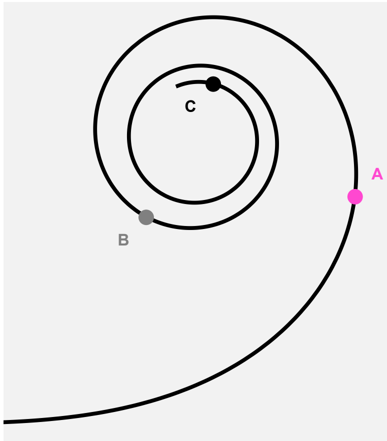
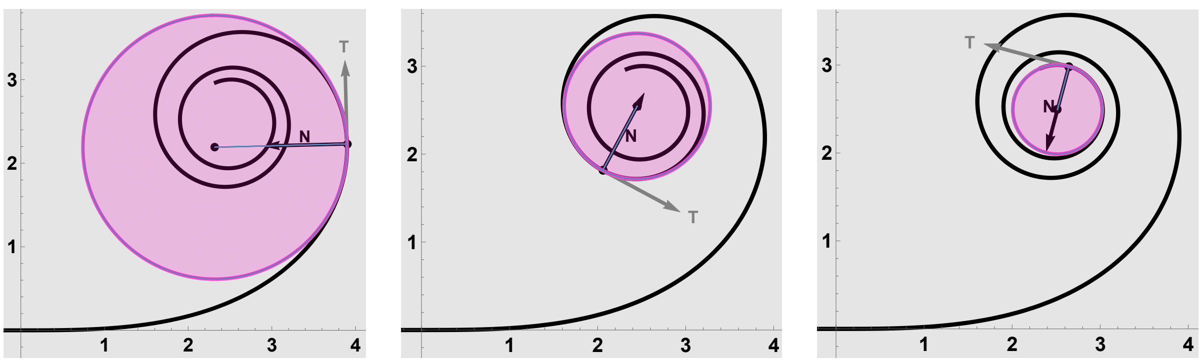


Figure 10: **The Fresnel Spiral**

Another example will illustrate more nuanced questions from differential geometry by analyzing the Fresnel ("fre-nel") spiral.

- **(Question)** Given curvature values at the three locations in Figure 10, can one predict the curvature at any other point?
- **(Answer)** The answer to this question begins with an analysis of osculating circles (see Figure 11) followed by a search for a pattern (see Figure 12).

Imagine, again, that a magnetic ball is attached to metal wire bent into the shape of the Fresnel spiral, and that you have propelled this ball along the wire by the application of some force. With your stop-watch in hand, imagine further that you collect data on the location (x, y) of the ball at a measured time. The data you collect is shown in Figure 12. Analyzing this data shows that after 3 units of time the magnetic ball was at point A, after 6 units of time it was at point B, and finally, after 9 units of time, it was at point C. Further analysis shows that, for every 1 unit of time, the curvature of the curve increases by 0.2. It follows that the curvature of the Fresnel spiral increases linearly with the amount of time spent following the spiral. That is, $\kappa_{\odot}(t) = 0.2t$ which predicts the curvature at a point (x, y) if one has measured the time it took to get to that point. For example, it takes $t = 1$ unit of time to reach the point $(x, y) \approx (1.60, 0.09)$ and therefore the curvature of the curve at this point is $0.2 \cdot 1 = 0.2$. [2]



(a) (Point A, $r_{\odot} \approx 1.65$) (b) (Point B, $r_{\odot} \approx 0.83$) (c) (Point C, $r_{\odot} \approx 0.55$)

Figure 11: **Curvature Values at A, B, and C.** Finding osculating circles at these three points and computing their reciprocals gives A ($\kappa_{\odot} \approx 0.61$), B ($\kappa_{\odot} \approx 1.21$), and C ($\kappa_{\odot} \approx 1.81$).

t	(x,y)	$r_{\odot}$	$k_{\odot}$
1	(1.60 0.09)	4.92	0.20
2	(3.07 0.67)	2.47	0.40
3	(3.89 1.99)	1.65	0.61
4	(3.28 3.38)	1.24	0.81
5	(1.83 3.19)	0.99	1.01
6	(2.06 1.81)	0.83	1.21
7	(3.20 2.48)	0.71	1.41
8	(2.05 2.92)	0.62	1.61
9	(2.73 2.00)	0.55	1.81
10	(2.33 2.97)	0.50	2.01

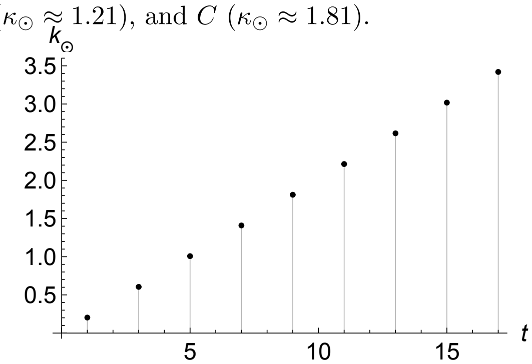
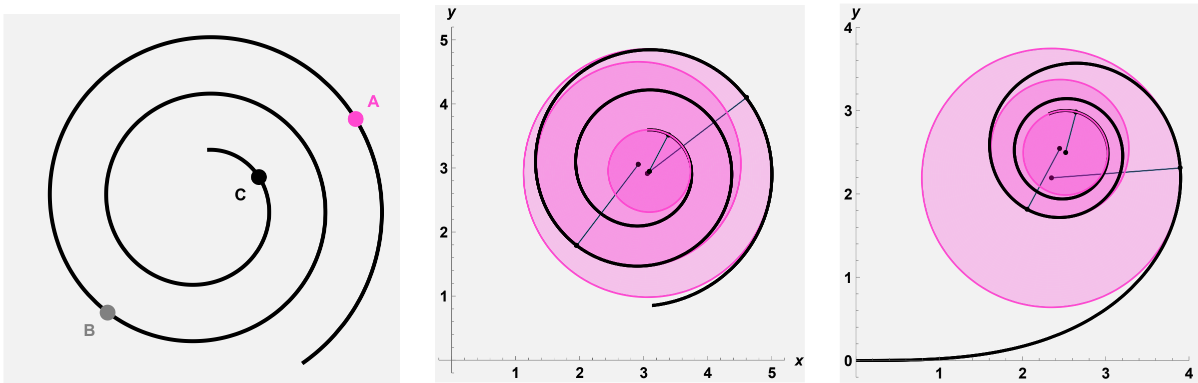


Figure 12: **Fresnel Spiral Data.**[2] The curvature of this Fresnel spiral grows linearly with time. Fitting a line to this data yields the formula $k_{\odot}(t) = 0.2 \cdot t$, which predicts curvature at other points on the curve.

[2] All values in this and subsequent tables are rounded to two decimal places.

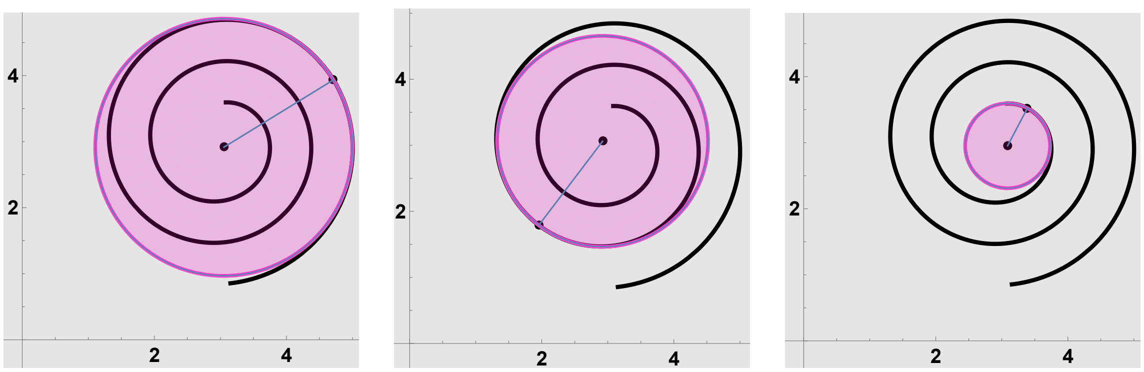
7 Example: The Archemedean Spiral

As a final example, we analyze the Archemedean spiral using the same techniques as in the previous example.



(a) The Archemedean Spiral. (b) Osculating Circles of Archemedean Spiral Are Decreasing in Radius. (c) Osculating Circles of Fresnel Spiral Are Also Decreasing in Radius.

Figure 13: **Archemedean Spiral vs. Fresnel Spiral.** Do the radii of the osculating circles decrease in the same way?



(a) (Point A, $r_\odot \approx 1.94$) (b) (Point B, $r_\odot \approx 1.60$) (c) (Point C, $r_\odot \approx 0.64$)

Figure 14: **Curvature Values at** A**,** B**, and** C**.** Finding osculating circles at these three points and computing their reciprocals gives A ($\kappa_\odot \approx 0.51$), B ($\kappa_\odot \approx 0.63$), and C ($\kappa_\odot \approx 1.56$).

t	(x,y)	$r_\odot$	$k_\odot$
1	(4.03 4.60)	1.90	0.53
3	(1.32 3.24)	1.70	0.59
5	(3.43 1.56)	1.50	0.67
7	(3.98 3.85)	1.30	0.77
9	(2.00 3.45)	1.10	0.91
11	(3.00 2.10)	0.89	1.12
13	(3.64 3.29)	0.69	1.44
15	(2.62 3.33)	0.49	2.04
17	(2.92 2.71)	0.29	3.48

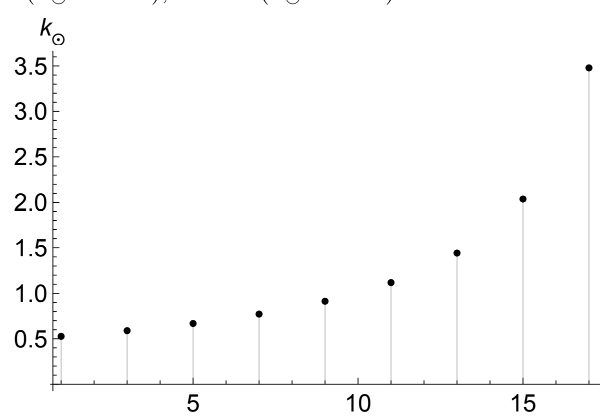
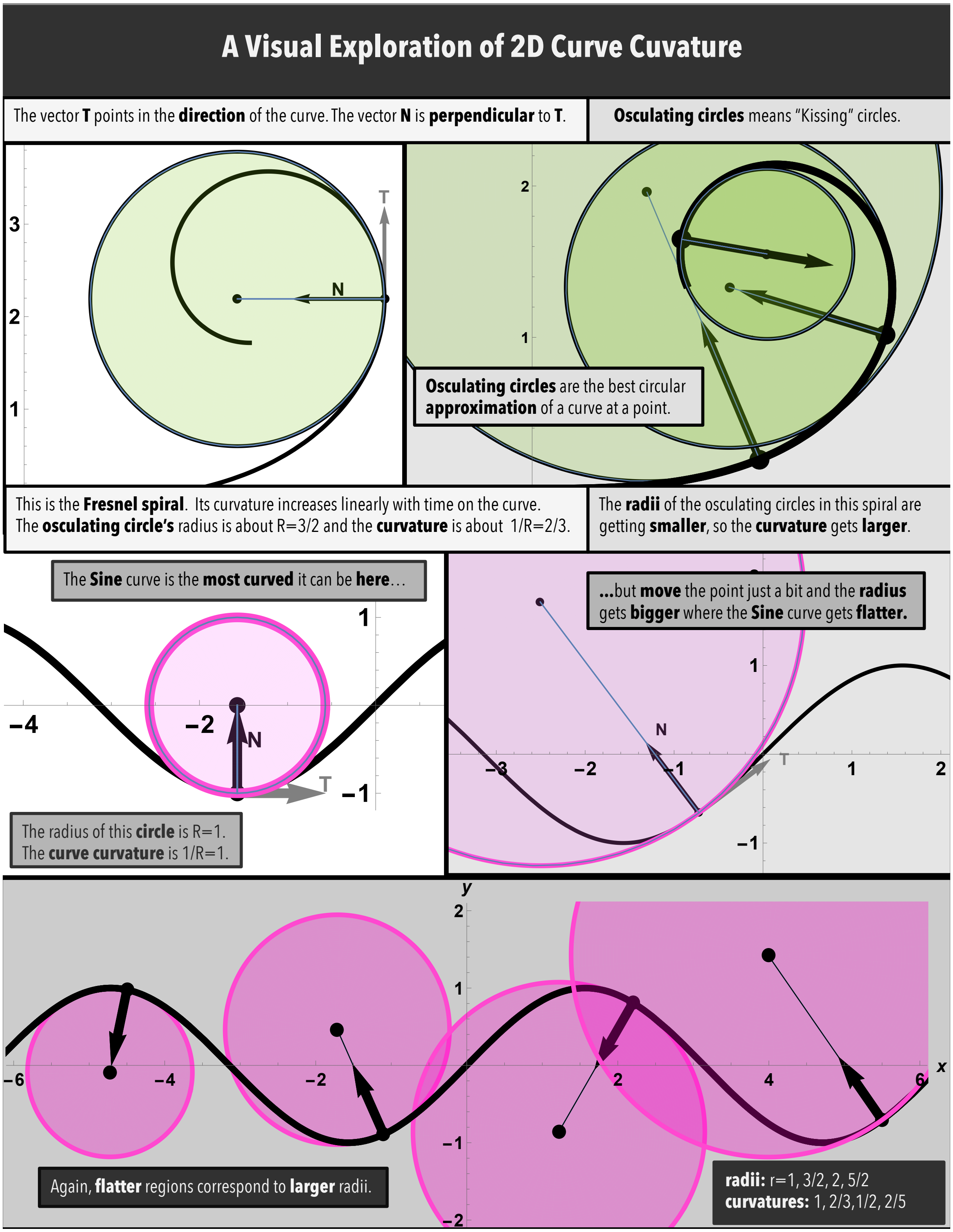


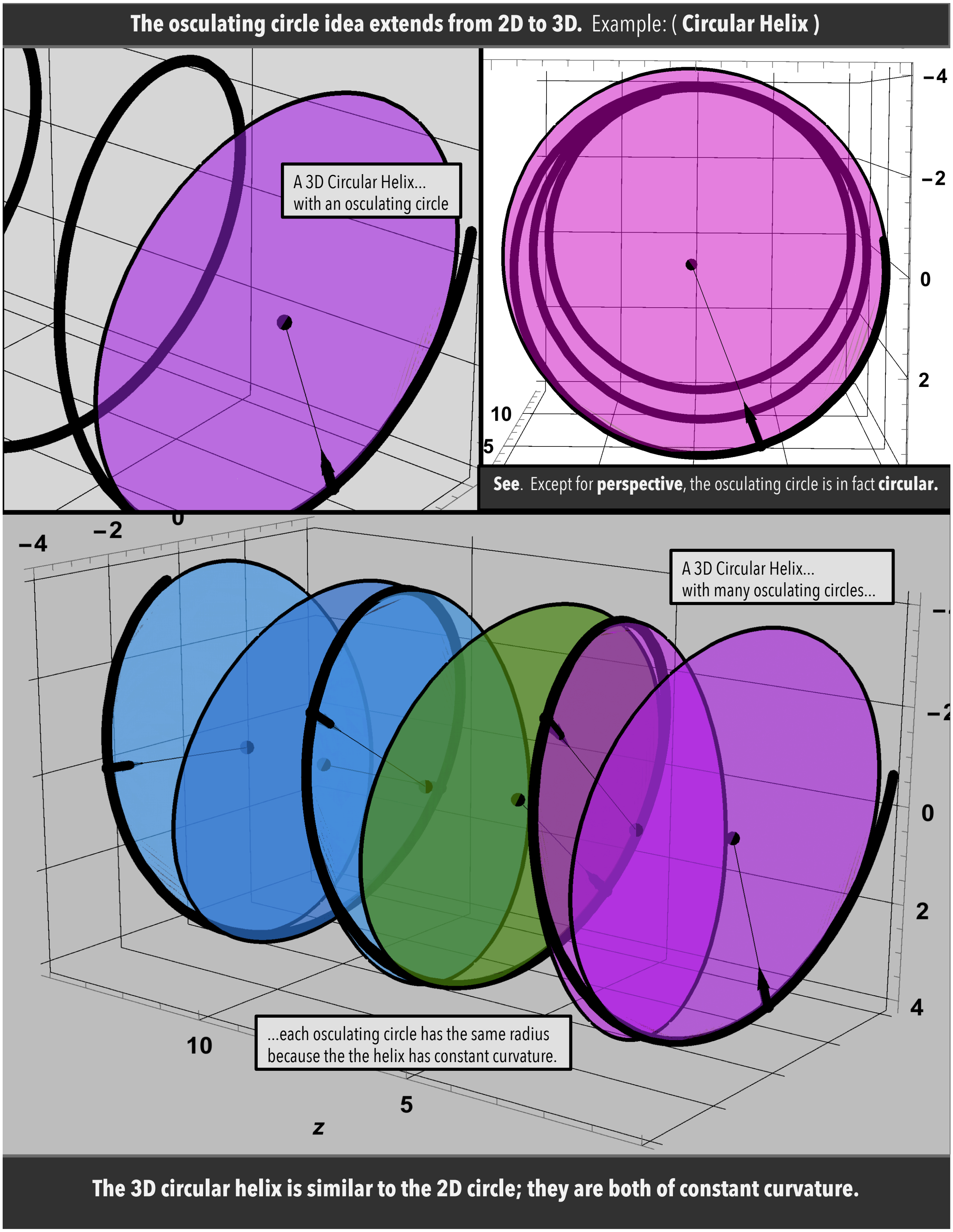
Figure 15: **Archemedean Spiral Data.** The curvature of this spiral grows non-linearly with time.

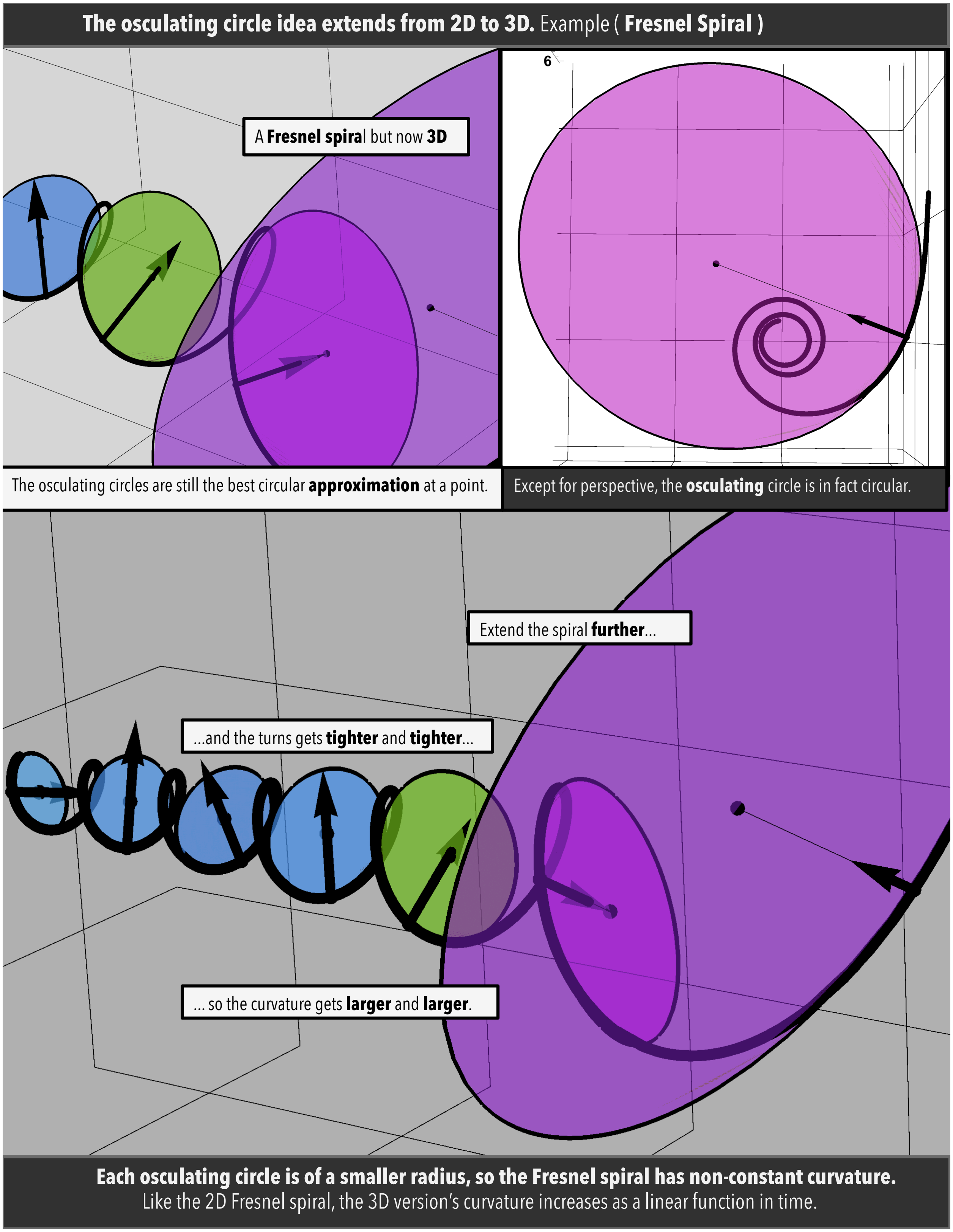
8 A Visual Summary of 2D Curve Curvature

Recall from [P.] (pg. 4), "Digital Natives are used to receiving information really fast. They like to parallel process and multi-task. They prefer their graphics before their text rather than the opposite". For the digital native, this page is a *visual interactive* summarizing the paper thus far.



9 Moving Into the Third Dimension




The osculating circle idea extends from 2D to 3D. Example (Fresnel Spiral)
A Fresnel spiral but now 3D
6
The osculating circles are still the best circular approximation at a point.
Except for perspective, the osculating circle is in fact circular.
Extend the spiral further...
...and the turns gets tighter and tighter...
... so the curvature gets larger and larger.
Each osculating circle is of a smaller radius, so the Fresnel spiral has non-constant curvature.
Like the 2D Fresnel spiral, the 3D version's curvature increases as a linear function in time.

10 A Brief Excursion into Calculus

In this final section we would like to offer a more technical approach to curvature for those interested. Curve curvature is often presented in multivariable calculus courses. Derivations and more lengthy discussions on the formulas presented here can be found in most standard calculus texts. See, for example, [L.] (Chapter 12) and [S.] (Chapter 1).

Imagine we have a curve and let $\mathbf{r}(t)$ (a vector–valued function) denote our position on that curve at some time t. Then $\mathbf{r}'(t)$ (the derivative) gives our velocity vector at time t. The length of the velocity vector, $|\mathbf{r}'(t)|$, gives our speed. If we assume that at no time are we stopped (i.e. $|\mathbf{r}'(t)| \neq 0$), then we can *normalize* our velocity vector and obtain $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$, the *unit tangent* vector at time t. Notice that $\mathbf{T}(t)$ has length $|\mathbf{T}(t)| = 1$ and points in our direction of motion.

Repeating this computation yields a vector valued function $\mathbf{T}'(t)$ (the change in the unit tangent). As long as the tangent changes (i.e. $\mathbf{T}'(t) \neq \mathbf{0}$), we can normalize this derivative as well and get $\mathbf{N}(t) = \mathbf{T}'(t)/|\mathbf{T}'(t)|$, the *unit normal* vector at time t. It turns out that, as long as $\mathbf{N}(t)$ is defined, $\mathbf{N}(t)$ and $\mathbf{T}(t)$ are perpendicular at every point in time t.

Recall that curvature was defined by analyzing the change in the tangent vector. The "official" formula for curvature is $\kappa(t) = |\mathbf{T}'(t)|/|\mathbf{r}'(t)|$. This is a measure of how fast the unit tangent is changing relative to our change in position along the curve. Notice that if we are traveling along a line, the tangent does not change so again $\kappa(t) = |\mathbf{T}'(t)|/|\mathbf{r}'(t)| = 0/|\mathbf{r}'(t)| = 0$. With a little more work, one can show (and often it is shown in a third semester calculus course) that $\kappa(t) = 1/r$ when our curve is a circle of radius r.

Finally, let us mention a special case of the curvature formula. When our curve is the graph of a function: $y = f(x)$, it is not difficult to show that the formula for curvature is $\kappa(x) = |f''(x)|/\left(1 + (f'(x))^2\right)^{3/2}$. Notice the appearance of the second derivative. This should not be too surprising given that the curvature is measuring how the unit tangent changes, and the tangent itself is a measure of change. Curvature, more or less, measures a change in the change.

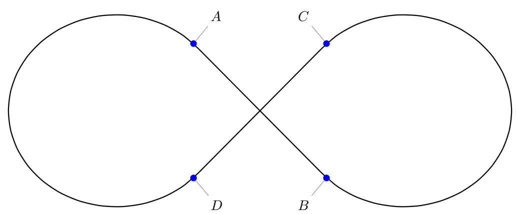
As a final example, when our curve is the parabola $y = x^2$, the above formula results in $\kappa(x) = 2/(1 + 4x^2)^{3/2}$. Notice that $\kappa(x)$ is largest when the denominator is minimized at $x = 0$. Therefore, the maximal curvature is $\kappa(0) = 2$ which occurs at the vertex of the parabola. This matches the curvature at point C in Figure 8. Plugging in $x \approx 0.60$ and $x \approx 1.77$ we can see that the formula matches the curvature in Figure 8 at the points B and A as well.

11 Exploration Questions

The six (6) exploration questions in this section could be assigned as homework problems or clasroom projects after a careful reading of the previous sections. These questions can be found as typeset worksheets "Curve Curvature Exploration Worksheets" with accompaning "Answer Key" online at [3]

#1 Racetrack Exploration

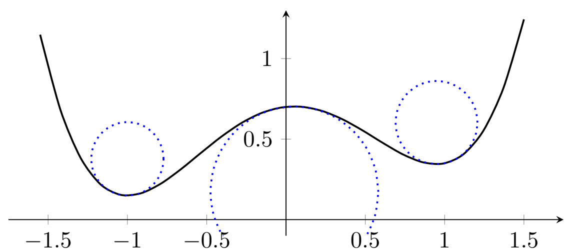
Consider the following racetrack (with a very unsafe intersection).



The track from A to B is a 60 foot long line segment. The track from B to C is a part of a circle of radius 30 feet and is 145 feet long. Then C to D is another 60 foot line segment followed by another 145 foot long circular portion with a 30 foot radius. Assuming that the circular portions of the track smoothly transition into the line segment portions, sketch a graph of the curvature of this track as a function of distance (in feet) from the starting line at A.

#2 Quartic Exploration

A plot of $f(x) = 0.5x^4 - 0.95x^2 + 0.10x + 0.7$ is shown below, sketch the curvature function of $f(x)$. What can we say about the curvature for large x inputs?

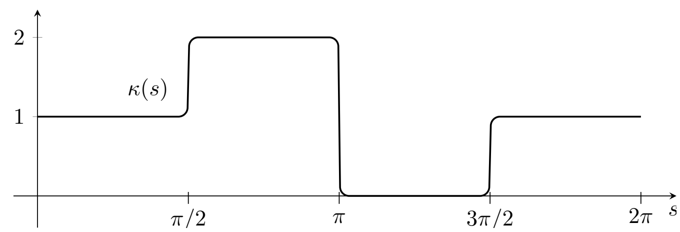


Note that the osculating circles at $x \approx -1.05$, 0.03, and 1.01 are shown as dotted circles, and their respective radii are approximately 0.228, 0.529, and 0.258.

[3] `http://mathsci2.appstate.edu/~osbornejm/InteractiveDemos/ExplorationWorksheets.pdf`

#3 Reverse Engineering Exploration

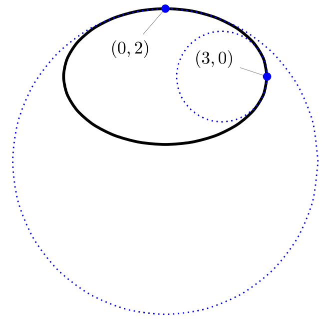
Sketch a curve which has a curvature function $\kappa(s)$ as pictured below.



Note that we have graphed the curvature as a function of arc length, s. So, for example, $\kappa(s) = 1$ for $0 \le s \le \pi/2$ means that the first piece of the curve, $\pi/2$ in length, has constant curvature $\kappa = 1$. *Hint: Think "6".*

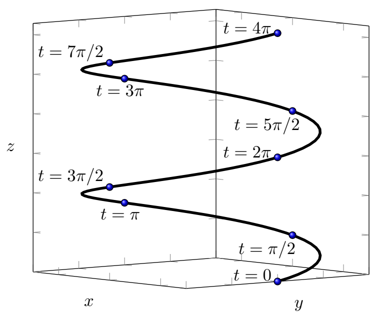
#4 Ellipse Exploration

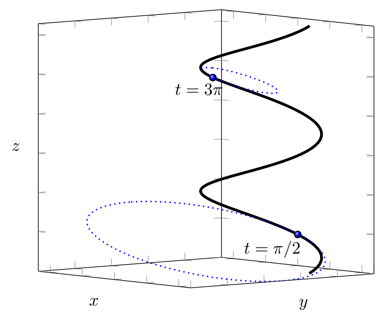
Consider the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ (which can be parameterized by $x(t) = 3\cos(t)$ and $y(t) = 2\sin(t)$). On the ellipse, the osculating circle at $(x, y) = (3, 0)$ (which corresponds with $t = 0$ in the given parameterization) has a radius of 4/3. The osculating circle at $(x, y) = (0, 2)$ (corresponding with $t = \pi/2$) has a radius of 9/2. Sketch the curvature function for the ellipse. What are the maximum and minimum curvatures of the ellipse?



#5 Elliptical Helix Exploration

Consider the elliptical helix parameterized by $x(t) = 3\cos(t)$, $y(t) = 2\sin(t)$, $z(t) = t$ shown below. On the elliptical helix, the osculating circles at $(x, y, z) = (0, 2, \pi/2)$ (corresponding to $t = \pi/2$) and $(x, y, z) = (-3, 0, 3\pi)$ (corresponding to $t = 3\pi$) are shown below. These circles radii are 5 and 5/3 respectively.





Sketch the curvature function for the elliptical helix. What are the maximum and minimum curvatures of the helix? *Hint: Consider exploring the 3D (circular) helix interactive demo before starting this particular exploration* [a].

[a] `https://mathsci2.appstate.edu/~osbornejm/InteractiveDemos.html`

#6 Symbolic Exploration

When our curve is the graph of a function $y = f(x)$, curvature is given by $\kappa(x) = \dfrac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$.

a. Use this formula to find $\kappa(x)$ when $f(x) = 0.5x^4 - 0.95x^2 + 0.1x + 0.7$ (the quartic from problem #2). Graph $\kappa(x)$ and compare with your previous sketch.

b. Consider a general polynomial, $g(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. As $x \to \pm\infty$, we know that the graph of $g(x)$ flattens out and looks more and more like a straight line. Using the formula for $\kappa(x)$, show that $\lim_{x \to \pm\infty} \kappa(x) = 0$, confirming our graphical observation.

c. (For those with knowledge of cross products:) When a curve is parameterized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, we have that $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$. Using the parameterization $\mathbf{r}(x) = x\mathbf{i} + f(x)\mathbf{j} + 0\mathbf{k}$ for $y = f(x)$, derive our special formula for curvature (stated above).

d. Using the above cross product formula and referring back to problems #4 and #5, compute the curvature of the ellipse (parameterized by $x(t) = 3\cos(t)$ and $y(t) = 2\sin(t)$) and elliptical helix (parameterized by $x(t) = 3\cos(t)$, $y(t) = 2\sin(t)$, $z(t) = t$). Graph these formulas for $\kappa(t)$ and compare with your previous sketches.

12 Conclusion and Invitation

Look back at what we have accomplished. We have investigated the sophisticated notion of curve curvature – a topic commonly reserved until a third-semester study of calculus – and extended this idea from 2D to 3D curves. And we accomplished this though simple geometric investigations using lines, vectors, and one simple ratio involving the radii of circles.

Along with investigating curve curvature, we hope that this paper provides an invitation to readers to investigate more advanced mathematics in the future. As seen, mathematics can be investigated formally or intuitively. A diversity of investigation techniques provides opportunities for many mathematical topics to be considered by a broad audience. Hopefully, the geometric methods introduced in this article further reveal how beautiful mathematics can be.

References

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[O.] O'Neill, B. *Elementary Differential Geometry*, Revised Second Ed., Academic Press (2006).

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[P.] Prensky. M. Digital Natives Digital Immigrants, **vol. 9. no. 5**, On the Horizon, MCB University Press (2001) (`https://www.marcprensky.com/writing/Prensky%20-%20Digital%20Natives,%20Digital%20Immigrants%20-%20Part1.pdf`).