Synthetic Division: Connecting with Other Mathematical Ideas

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Abstract

Synthetic division, as developed by Ruffini in 1804, was limited to division of a polynomial by a linear polynomial factor in the form $x - c$. Connected to Ruffini's method, in the early 1800's, Horner developed techniques for finding roots and determining the derivatives of polynomials. Additionally, Horner expanded Ruffini's method of synthetic division so that a polynomial could be divided by polynomials of higher degree than just 1. Some high school and college students have used synthetic division to divide by a linear polynomial factor. However, few students may know why and how this process works. The increasing use of computers and calculators with algebraic operating systems may further hide the beauty of synthetic division. This paper: (A) describes how synthetic division can be used in the contexts of integer, rational, real, and complex divisors; (B) investigates Horner's method of synthetic division to divide by polynomials of any degree; and (C) makes connections with the Remainder Theorem and the Zero Product Property. This paper provides the reader with students investigations and an applet for performing polynomial division.

1 Introduction & Leading Questions

We introduce this discussion of synthetic division through some investigatory questions. (Answers to these questions appear below.[1])

Consider the polynomial $P(x) = 2x^6 + 3x^5 - x^4 - 3x^3 - 5x^2 - 6x - 2$.

1. What are all possible rational roots for $P(x)$? (Hint: consider the Rational Roots Theorem.)

2. What are all the possible real roots for $P(x)$?

3. Is it possible for $P(x)$ to have complex roots? Why or why not?

4. What can you deduce from the following synthetic division?

$$
\begin{array}{r|rrrrr}
 & 2 & -3 & 4 & -5 & 6 \\
2 & \downarrow & 4 & 2 & 12 & 14 \\
\hline
 & 2 & 1 & 6 & 7 & \boxed{20}
\end{array}
$$

5. What can you deduce from the following synthetic division?

$$
\begin{array}{r|rrrrrrr}
 & 2 & 3 & -1 & -3 & -5 & -6 & -2 \\
i & \downarrow & 2i & -2+3i & -3-3i & 3-6i & 6-2i & 2 \\
\hline
 & 2 & 3+2i & -3+3i & -6-3i & -2-6i & -2i & \boxed{0}
\end{array}
$$

(Web Example #2 on https://billcookmath.com/sage/algebra/Horners_method.html.)

6. Look at the entire multi-step synthetic division. What does it tell us?

$$
\begin{array}{r|rrrrrrr}
 & 2 & 3 & -1 & -3 & -5 & -6 & -2 \\
i & \downarrow & 2i & -2+3i & -3-3i & 3-6i & 6-2i & 2 \\
\hline
 & 2 & 3+2i & -3+3i & -6-3i & -2-6i & -2i & \boxed{0} \\
-i & \downarrow & -2i & -3i & 3i & 6i & 2i & \\
\hline
 & 2 & 3 & -3 & -6 & -2 & \boxed{0} & \\
\sqrt{2} & \downarrow & 2\sqrt{2} & 4+3\sqrt{2} & 6+\sqrt{2} & 2 & & \\
\hline
 & 2 & 3+2\sqrt{2} & 1+3\sqrt{2} & \sqrt{2} & \boxed{0} & & \\
-\sqrt{2} & \downarrow & -2\sqrt{2} & -3\sqrt{2} & -\sqrt{2} & & & \\
\hline
 & 2 & 3 & 1 & \boxed{0} & & & \\
-\frac{1}{2} & \downarrow & -1 & -1 & & & & \\
\hline
 & 2 & 2 & \boxed{0} & & & & \\
-1 & \downarrow & -2 & & & & & \\
\hline
 & 2 & \boxed{0} & & & & &
\end{array}
$$

[1] **Solutions to the Introductory Questions**

1. The rational root theorem (source) says that the only possible rational roots are $x = \pm\dfrac{\text{factors of constant term}}{\text{factors of leading term}}$. Thus the possible rational roots are $x = -1, -2, 1, 2$.
2. To determine the possible real roots, you could use Descartes rule of signs (source). Because $P(x)$ is a sixth degree polynomial, we know that there are either 6,4,2, or 0 real roots since complex roots come in pairs. Graph $P(x)$ with Desmos.
3. $P(x)$ may have complex roots because the Fundamental Theorem of Algebra gives 6 roots and we know there are at most 4 rational roots, therefore there must be at least 2 real roots or a complex conjugate pair of roots.
4. The synthetic division demonstrates that $x-2$ is not a factor of $P(x)$ and the division $\frac{2x^4-3x^3+4x^2-5x+6}{x-2}$ produces a remainder of $\frac{20}{x-2}$. Additionally it reveals that $P(2) = 20$.
5. The synthetic division shows that $x-i$ is a factor of $P(x)$ because the remainder term is 0, and thus $P(i) = 0$.
6. The entire multi-step synthetic division shows that $P(x) = 2(x+1)(x+\frac{1}{2})(x+\sqrt{2})(x-\sqrt{2})(x+i)(x-i)$.

These initial questions lead to both observations and additional questions. First, most high school and college students have only used synthetic division to divide by a linear factor involving an integer (e.g., dividing by $x-2$). In questions 4 and 5, we see that synthetic division can be used for integer, rational, irrational, real and even complex roots.

Second, since in questions 5 and 6 each line of synthetic division results with a remainder of zero (we will later consider the Remainder Theorem), we know that $P(x) = 2x^6+3x^5-x^4-3x^3-5x^2-6x-2 = 2(x+1)(x+\frac{1}{2})(x+\sqrt{2})(x-\sqrt{2})(x+i)(x-i)$ which can be written as $2(x^2+1)(x^2-2)(x+\frac{1}{2})(x+1)$.

Before moving on, let us consider one more possibly unexpected example:[2]

$$\frac{x^3+x^2-x+15}{x^2-2x+5} = \frac{x^3+x^2-x+15}{\Big(x-(1-2i)\Big)\Big(x-(1+2i)\Big)}$$

$$
\begin{array}{r|rrrr}
 & 1 & 1 & -1 & 15 \\
1-2i & \downarrow & 1-2i & -2-6i & -15 \\
\cline{2-5}
 & 1 & 2-2i & -3-6i & \boxed{0} \\
1+2i & \downarrow & 1+2i & 3+6i & \\
\cline{2-4}
 & 1 & 3 & \boxed{0} &
\end{array}
$$

Now we can see that synthetic division can be performed with complex roots. But we should still note that the complex number $1+2i$ is a single complex coefficient and not two coefficients.

With these observations now in place, in the following sections we can investigate how synthetic division works and whether synthetic division is limited to dividing by linear factors.

2 Investigating and Extending Synthetic Division

Students are often introduced to synthetic division as shorthand notation for dividing a polynomial by a linear factor [1]. Figure 1 shows an example of long division ($2x^4-3x^3+4x^2-5x+6$ divided by $x-2$) next to its associated form of synthetic division.[3] Comparing the two forms illuminates from whence the values in the synthetic division evolve.

[2] Web Examples #3, #3A, and #3B on https://billcookmath.com/sage/algebra/Horners_method.html

[3] Web Example #4 on https://billcookmath.com/sage/algebra/Horners_method.html

$$\begin{array}{r|rrrrrl}
 & & 2x^3 & +1x^2 & +6x & +7 & R20 \\
\hline
x-2 & 2x^4 & -3x^3 & +4x^2 & -5x & +6 & \\
 & -(2x^4 & -4x^3) & \downarrow & \downarrow & \downarrow & \\
 & & +1x^3 & +4x^2 & \downarrow & \downarrow & \\
 & & -(x^3 & -2x^2) & \downarrow & \downarrow & \\
 & & & 6x^2 & -5x & \downarrow & \\
 & & & -(6x^2 & -12x) & \downarrow & \\
 & & & & 7x & +6 & \\
 & & & & -(7x & -14) & \\
 & & & & & 20 &
\end{array}$$

(a) long division

$$\begin{array}{r|rrrrrl}
 & 2 & -3 & 4 & -5 & 6 & \\
2 & \downarrow & 4 & 2 & 12 & 14 & \\
\hline
 & 2 & 1 & 6 & 7 & \boxed{20} & = f(2)
\end{array}$$

(b) synthetic division

Figure 1: Side-by-side example showing both long division and synthetic division

Thus, as we will later see through the Remainder Theorem, $f(2) = 20$, and $f(x) = (2x^3 + x^2 + 6x + 7) \cdot (x - 2) + 20$.

Below are two figures. One demonstrates a looping algorithm which takes place in the process of long division. The other demonstrates an analogous looping procedure in the process of synthetic division. With some observation, one can see that these looping techniques share much in common.

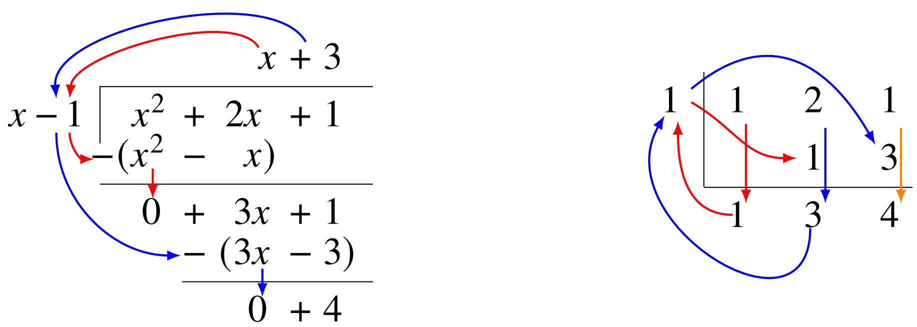

Figure 2: Looping algorithms for long division and synthetic division.

Possibly to the surprise of many students, this method can also be used to divide by linear factors whose leading coefficient is other than 1. For example,[4] let us consider $3x^3+2x-4$ divided by $2x-3$. First, notice that we can write the factor $2x - 3$ into the form $x - \frac{3}{2}$. Let us first perform synthetic division by $x - \frac{3}{2}$, since this seems to be in the more familiar form with a leading coefficient of 1.

$$\begin{array}{r|rrrr}
 & 3 & 0 & 2 & -4 \\
\frac{3}{2} & \downarrow & \frac{9}{2} & \frac{27}{4} & \frac{105}{8} \\
\hline
 & 3 & \frac{9}{2} & \frac{35}{4} & \boxed{\frac{73}{8}}
\end{array}$$

[4] Web Example #5 on `https://billcookmath.com/sage/algebra/Horners_method.html`

Let us now do the synthetic division in a slightly modified form by $2x-3$.

$$\begin{array}{r|cccc}
 & 3 & 0 & 2 & -4 \\
2 & \downarrow & \frac{9}{2} & & \\
3 & \downarrow & \downarrow & \frac{27}{4} & \\
\hline
 & \frac{3}{2} & \frac{9}{4} & \frac{35}{8} & \boxed{\frac{73}{8}}
\end{array}$$

The details of this process are explained in Section 2.1.

STOP. Let's be certain to understand the work in the last example; this technique leads to foundational ideas for the remainder of this paper. First, notice that division by $x-\frac{3}{2}$ and $2x-3$ produced the same results. Second, notice that in all previous examples of synthetic division prior to division by $2x-3$, only the constant term is on the outside left. In the last example with division by $2x-3$, we have two terms on the outside left.

The question which now naturally arises is, "If we can have either one or two terms on the outside left of the synthetic division, might we be able to have more?" And, if more, is there a limit? And, if we have a on the outside left, that means we are dividing by $x-a$. If we have a and b on the outside left, we are dividing by $ax-b$. But what does it mean if we have a, b, and c on the outside left, what does this mean that we are dividing by: ax^2+bx+c, ax^2-bx+c, ax^2-bx-c, or some other form?

Excuse me? Did we just imply that we can perform synthetic division by a linear or quadratic polynomial, and, by implication, possibly any real polynomial of any degree? Let us examine one example of polynomial division by a sparse[5] quartic polynomial. This example uses multiple coefficients of 0 and 1 in the divisor to make the process more transparent and illuminating. We encourage the reader to carefully examine this example to determine from where values in the synthetic division evolve. Consider $\dfrac{7x^6+6x^5+4x^3-1}{x^4+1}$, leading to the following synthetic division.[6]

$$\begin{array}{r|ccc|cccc}
 & 7 & 6 & 0 & 4 & 0 & 0 & -1 \\
1 & \downarrow & 0 & 0 & 0 & -7 & & \\
0 & \downarrow & \downarrow & 0 & 0 & 0 & -6 & \\
0 & \downarrow & \downarrow & \downarrow & 0 & 0 & 0 & 0 \\
0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
-1 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\hline
 & 7 & 6 & 0 & \boxed{4 \quad -7 \quad -6 \quad -1}
\end{array}$$

2.1 Expanding the Process in Order to Understand the Compressed Process

In the following example (Figure 2.1), each step in the expanded synthetic division technique is taken one at a time. While this makes the example quite lengthy, it should further develop the notion of the technique being used. Also, within the example, products and sums are shown to better demonstrate

[5] a polynomial with a relatively high number of zero coefficients

[6] Web Example #6 on https://billcookmath.com/sage/algebra/Horners_method.html

the process. Our example[7] is:

$$\frac{6x^5 - 5x^4 + 4x^3 - 3x^2 + 2x - 1}{2x^2 - x + 3}.$$

First set up the synthetic division by putting the coefficients of the divisor across the top (in blue) and the coefficients of the dividend vertically (in red). If the numerator is degree n and the denominator is degree m, then, including the column to the left for the divisor, there will be $n + 1$ columns. Including the top row for the dividend and the bottom row for the solution or quotient, there will be $m + 2$ rows. Additionally, the vertical line separating the quotient from the remainder will be after the $(n - m + 1)th$ number across the top. Notice that the leading coefficient of the dividend remains the same (that is 2) while the signs of the non-leading coefficients are changed (to 1 and -3). When you bring down the **6**, the leading coefficient of the dividend, you divide by the leading coefficient of the divisor, **2**. Next, multiply the **3** by the non-leading coefficients of the dividend. Then add down the next column and divide by the dividend's leading coefficient again. Continue this process for a total of 4 times, and thus,

$$\frac{\mathbf{6}x^5 - \mathbf{5}x^4 + \mathbf{4}x^3 - \mathbf{3}x^2 + \mathbf{2}x - \mathbf{1}}{\mathbf{2}x^2 - \mathbf{1}x + \mathbf{3}} = \mathbf{3}x^3 - \mathbf{1}x^2 - \mathbf{3}x - \frac{\mathbf{3}}{\mathbf{2}} + \frac{\frac{\mathbf{19}}{\mathbf{2}}x + \frac{\mathbf{7}}{\mathbf{2}}}{\mathbf{2}x^2 - \mathbf{1}x + \mathbf{3}}.$$

$$\begin{array}{r|rrrr|rr} & \mathbf{6} & \mathbf{-5} & \mathbf{4} & \mathbf{-3} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \downarrow & & & & & \\ \mathbf{1} & \downarrow & & & & & \\ \mathbf{-3} & \downarrow & & & & & \\ & \downarrow & & & & & \\ \hline & \frac{\mathbf{6}}{\mathbf{2}} = \mathbf{3} & & & & & \end{array}$$

(a) Set up & Step 1

$$\begin{array}{r|rrrr|rr} & \mathbf{6} & \mathbf{-5} & \mathbf{4} & \mathbf{-3} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \downarrow & (\mathbf{3})(\mathbf{1}) & (\mathbf{3})(\mathbf{-3}) & & & \\ \mathbf{1} & \downarrow & & & & & \\ \mathbf{-3} & \downarrow & & & & & \\ & \downarrow & & & & & \\ \hline & \mathbf{3} & & & & & \end{array}$$

(b) Multiply **3** by **1** and **-3**

$$\begin{array}{r|rrrr|rr} & \mathbf{6} & \mathbf{-5} & \mathbf{4} & \mathbf{-3} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \downarrow & \mathbf{3} & \mathbf{-9} & & & \\ \mathbf{1} & \downarrow & \downarrow & & & & \\ \mathbf{-3} & \downarrow & \downarrow & & & & \\ & \downarrow & \downarrow & & & & \\ \hline & \mathbf{3} & \frac{\mathbf{-5+3}}{\mathbf{2}} = \mathbf{-1} & & & & \end{array}$$

(c) Add down column 2 and divide by **2**

$$\begin{array}{r|rrrr|rr} & \mathbf{6} & \mathbf{-5} & \mathbf{4} & \mathbf{-3} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \downarrow & \mathbf{3} & \mathbf{-9} & & & \\ \mathbf{1} & \downarrow & \downarrow & (\mathbf{-1})(\mathbf{1}) & (\mathbf{-1})(\mathbf{-3}) & & \\ \mathbf{-3} & \downarrow & \downarrow & & & & \\ & \downarrow & \downarrow & & & & \\ \hline & \mathbf{3} & \mathbf{-1} & & & & \end{array}$$

(d) Multiply **-1** by **1** and **-3**

[7]Web Example #7 on https://billcookmath.com/sage/algebra/Horners_method.html

$$
\begin{array}{r|rrrr|rr}
 & 6 & -5 & 4 & -3 & 2 & -1 \\
2 & \downarrow & 3 & -9 & & & \\
1 & \downarrow & \downarrow & -1 & 3 & & \\
-3 & \downarrow & \downarrow & \downarrow & & & \\
 & \downarrow & \downarrow & \downarrow & & & \\
\hline
 & 3 & -1 & \frac{4-9-1}{2} = -3 & & &
\end{array}
$$

(e) Add down column 3 and divide by 2

$$
\begin{array}{r|rrrr|rr}
 & 6 & -5 & 4 & -3 & 2 & -1 \\
2 & \downarrow & 3 & -9 & & & \\
1 & \downarrow & \downarrow & -1 & 3 & & \\
-3 & \downarrow & \downarrow & \downarrow & (-3)(1) & (-3)(-3) & \\
 & \downarrow & \downarrow & \downarrow & & & \\
\hline
 & 3 & -1 & -3 & & &
\end{array}
$$

(f) Multiply -3 by 1 and -3

$$
\begin{array}{r|rrrr|rr}
 & 6 & -5 & 4 & -3 & 2 & -1 \\
2 & \downarrow & 3 & -9 & & & \\
1 & \downarrow & \downarrow & -1 & 3 & & \\
-3 & \downarrow & \downarrow & \downarrow & -3 & 9 & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & & \\
\hline
 & 3 & -1 & -3 & \frac{-3+3-3}{2} = -\frac{3}{2} & &
\end{array}
$$

(g) Add down column 4 and divide by 2

$$
\begin{array}{r|rrrr|rr}
 & 6 & -5 & 4 & -3 & 2 & -1 \\
2 & \downarrow & 3 & -9 & & & \\
1 & \downarrow & \downarrow & -1 & 3 & & \\
-3 & \downarrow & \downarrow & \downarrow & -3 & 9 & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & (-\frac{3}{2})(1) & (-\frac{3}{2})(-3) \\
\hline
 & 3 & -1 & -3 & -\frac{3}{2} & &
\end{array}
$$

(h) Multiply $-\frac{3}{2}$ by 1 and -3

$$
\begin{array}{r|rrrr|rr}
 & 6 & -5 & 4 & -3 & 2 & -1 \\
2 & \downarrow & 3 & -9 & & & \\
1 & \downarrow & \downarrow & -1 & 3 & & \\
-3 & \downarrow & \downarrow & \downarrow & -3 & 9 & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & -\frac{3}{2} & \frac{9}{2} \\
\hline
 & 3 & -1 & -3 & -\frac{3}{2} & &
\end{array}
$$

(i) Simplify row 5

$$
\begin{array}{r|rrrr|rr}
 & 6 & -5 & 4 & -3 & 2 & -1 \\
2 & \downarrow & 3 & -9 & & & \\
1 & \downarrow & \downarrow & -1 & 3 & & \\
-3 & \downarrow & \downarrow & \downarrow & -3 & 9 & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & -\frac{3}{2} & \frac{9}{2} \\
\hline
 & 3 & -1 & -3 & -\frac{3}{2} & \frac{19}{2} & \frac{7}{2}
\end{array}
$$

(j) Add down columns 5 & 6

Figure 3: Detailed example of synthetic division with a nonlinear dividend

Before formalizing this process in the next section, we provide one more example for the reader to investigate. We recommend that you try this initially on your own and then look at the pattern and solution in the figure to ensure that you did the process correctly and got the right answer. Try:[8] $\dfrac{x^4 + x^2 - x + 5}{x^2 + 2x + 3}$.

$$
\begin{array}{r|rrr|rr}
 & 1 & 0 & 1 & -1 & 5 \\
1 & \downarrow & -2 & -3 & & \\
-2 & \downarrow & \downarrow & 4 & 6 & \\
-3 & \downarrow & \downarrow & \downarrow & -4 & -6 \\
\hline
 & 1 & -2 & 2 & \boxed{1} & \boxed{-1}
\end{array}
$$

Additional examples can be seen in [3]. and [8]

In the appendix, we provide problems for student investigations regarding synthetic division. These problems are developed to both deepen the student's understanding of synthetic division and to help them better understand the material in this paper.

[8]Web Example #8 on https://billcookmath.com/sage/algebra/Horners_method.html

3 Horner's Method of Synthetic Division and an Application

Without specifically naming such, we have already investigated Horner's Method of synthetic division. Notably, beyond traditional synthetic division, Horner's Method allows division by polynomials of any degree. However, Horner's Method has a number of applications such as root finding [4, 6, 7] and determining derivatives [5, 2] . We will consider only polynomial division here.

For any polynomial $P(x)$, there are a calculable number of operations necessary to evaluate $P(x)$ at some value r, of $P(r)$. Let us first cleverly factor a given polynomial using Horner's Form. We can see that

$$\begin{aligned} q(x) &= x^5 - 2x^4 - x^3 + 2x^2 - 2x + 4 \\ &= ((((x-2)\cdot x - 1)\cdot x + 2)\cdot x - 2)\cdot x + 4 \end{aligned} \tag{1}$$

To compute $q(x)$ for a specific x using the first form of $q(x)$ requires 5 additions with 13 multiplications for a total of 18 operations. Horner's factored form requires 5 additions and 4 multiplications for a total of 9 operations, significantly reducing the number of operations.

In general, for a fifth degree polynomial:

$$\begin{aligned} p(x) &= a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \\ &= ((((a_5x + a_4)\cdot x + a_3)\cdot x + a_2)\cdot x + a_1)\cdot x + a_0 \end{aligned}$$

Evaluating $p(r)$ using Horner's Form is

$$p(r) = ((((a_5r + a_4)\cdot r + a_3)\cdot r + a_2)\cdot r + a_1)\cdot r + a_0$$

We can express Horner's form schematically by

$$\begin{array}{c|cccccc} p & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\ \hline r & \downarrow & r\cdot a_5 & r\cdot A_4 & r\cdot A_3 & r\cdot A_2 & r\cdot A_1 \\ \hline & a_5 & A_4 = r\cdot a_5 + a_4 & A_3 = r\cdot A_4 + a_3 & A_2 = r\cdot A_3 + a_2 & A_1 = r\cdot A_2 + a_1 & \boxed{A_0 = r\cdot A_1 + a_0} \end{array}$$

Then

$$A_0 = p(r)$$

For example, evaluating q from (1) above at $r = -1$ is

$$\begin{array}{c|cccccc} q & 1 & -2 & -1 & 2 & -2 & 4 \\ \hline -1 & \downarrow & -1 & +3 & -2 & 0 & +2 \\ \hline & 1 & -3 & +2 & 0 & -2 & \boxed{A_0 = 6} \end{array}$$

Thus, $q(-1) = 6$.
Therefore, Horner's techniques surpass simply performing synthetic division.

In the appendix, we provide problems for student investigations regarding Horner's Method. These problems are developed to both deepen the student's understanding of Horner's Method and to help them better understand the material in this paper.

4 Remainder Theorem & Extension (nonlinear factors)

One of the possible beautiful connections that the reader could have realized from the previous student investigations is in respect to the wonderful Remainder Theorem [9]. (A more detailed, axiomatic, and mathematically precise development of the Remainder Theorem is provided in a footnote.[9] Readers can investigate the mathematics and style of this footnote and glean much regarding the beauty and intricacy of the mathematics being considered in this section of the paper.) This theorem states that when you divide a polynomial $P(x)$ by a linear polynomial, you produce another polynomial $Q(x)$ and a remainder R (a constant polynomial). This is written as

$$\frac{P(x)}{x-c} = Q(x) + \frac{R}{x-c}$$

OR

$$P(x) = Q(x)(x-c) + R.$$

In a number of ways, the Remainder Theorem has been a central connecting fiber to almost all which has preceded. Indeed, the remainder R is a deeply meaningful notion. When $R = 0$, it indicates that $x - c$ is a factor of $P(x)$. This can be seen as

$$\frac{P(x)}{x-c} = Q(x) + \frac{0}{x-c} = Q(x) + 0 = Q(x)$$

OR

$$P(x) = Q(x)(x-c).$$

9

Theorem 1 (Division Algorithm) *Let $n, d \in \mathbb{Z}$ where $d > 0$. Then there exist unique integers q, the* quotient*, and r, the* remainder*, such that*

$$n = q \cdot d + r$$

where $0 \leq r < d$. And d is a factor *of n if and only if $r = 0$*

The *division algorithm* extends to polynomials.

Definition 2 (Polynomials over a Field) *Let k be a field. For example: k could be $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. Then*

$$k[x] = \{p(x) \,|\, p \text{ is a polynomial in } x \text{ with coefficients in } k\}.$$

Theorem 3 (Division Algorithm for Polynomials) *Let k be a field. Let $f(x)$ and $d(x)$ be in $k[x]$ with $d(x) \neq 0$. Then there are unique polynomials $q(x)$, the* quotient*, and $r(x)$, the* remainder*, in $k[x]$ with*

$$f(x) = q(x) \cdot d(x) + r(x)$$

where either $r(x) = 0$ or $\deg(r) < \deg(d)$.

Note that the degree of the zero polynomial is undefined.

Corollary 4 (Remainder Theorem) *Let k be a field with $f(x) \in k[x]$ and $a \in k$. Then there is a (unique) $q(x) \in k[x]$ such that*

$$f(x) = q(x) \cdot (x-a) + f(a).$$

Corollary 5 (Factor Theorem) *Let k be a field with $f(x) \in k[x]$ and $a \in k$. Then a is a root of f iff $(x-a)$ is a factor of $f(x)$; i.e., there is a $q(x) \in k[x]$ such that*

$$f(x) = q(x) \cdot (x-a).$$

This alone is meaningful, and we saw this repeatedly in earlier examples of synthetic division, where the last term in the synthetic division $P(x) \div (x - c)$ was 0. From that, we knew that $x - c$ was a factor of $P(x)$.

However, what does the remainder tell us when $R \neq 0$? Well, it tells us at least two things. First, $x - c$ is not a factor of $P(x)$. Second, and this is the same whether or not $R = 0$, we always have $R = P(c)$, or R is the y-value of $P(x)$ when $x = c$. In other words,

$$P(c) = Q(c)(c - c) + R = R.$$

This now brings us to a wonderful intersection of some previously posed ideas. Remember that Horner's Method of synthetic division allowed us to divide a real polynomial by any real polynomial of lower degree, let's say $D(x)$, and not simply by a linear polynomial in the form $x - c$. So, we need tools which allow $D(x)$ to be *any* polynomial.

To our rescue is the Fundamental Theorem of Algebra, which states that every real polynomial can be factored over the reals into a product of linear and irreducible quadratic factors. We can now consider an extension of the Remainder Theorem: Let $I(x)$ be a real irreducible quadratic polynomial, then

$$\frac{P(x)}{I(x)} = Q(x) + \frac{R(x)}{I(x)}$$

OR

$$P(x) = Q(x)I(x) + R(x)$$

For example,[10] let $P(x) = x^4 - x^2 - 2x - 1$ and the irreducible quadratic $I(x) = x^2 + x + 1$, then

$$\frac{P(x)}{I(x)} = \frac{x^4 - x^2 - 2x - 1}{x^2 + x + 1} = x^2 - x - 1 + \frac{0}{x^2 + x + 1}.$$

Therefore, since $R(x) = 0$, $I(x)$ is a factor of $P(x)$.

If we alter this example slightly[11] and let $G(x) = x^4 + x^2 - x + 5$ and the irreducible quadratic $I(x) = x^2 + x + 1$, then

$$\frac{G(x)}{I(x)} = \frac{x^4 + x^2 - x + 5}{x^2 + x + 1} = x^2 - x + 1 + \frac{-x + 4}{x^2 + x + 1}.$$

In this case, since $R(x)$ is not identically 0, $I(x)$ is not a factor of $G(x)$.

Previously, we saw that $P(c) = R$ when dividing by the linear expression $x - c$. Dividing by a quadratic $q(x)$ gives two possibilities when q has real roots. If $q(x)$ has roots $x = r_1$ and r_2, then $R = R(x)$, and we (still) have $P(r_1) = R(r_1)$ and $P(r_2) = R(r_2)$. When the quadratic $q(x)$ has no real roots, i.e., is irreducible over the reals, then what would happen if we evaluated at a complex root $r = a \pm bi$ of $q(x)$? That is, is there a relation between $P(a \pm bi)$ and $R(a \pm bi)$? Using the Remainder Theorem easily answers this question.

In the appendix, we provide problems for student investigations regarding the Remainder Theorem. These problems are developed to both deepen the student's understanding of the Remainder Theorem and to help them better understand the material in this paper.

[10] Web Example #9 on https://billcookmath.com/sage/algebra/Horners_method.html
[11] Web Example #10 on https://billcookmath.com/sage/algebra/Horners_method.html

5 Conclusion

Explaining, expanding upon, and connecting mathematical ideas is central to student learning. In this paper we have explained the process of synthetic division commonly seen in high school. We have investigated why it works and how it connects to polynomial long division. Commonly seen synthetic division was then extended to help students see that synthetic division could consider linear factors in the form $x - c$, where c is an integer, rational, irrational, and even complex number. We then expanded upon synthetic division by considering Horner's method for dividing by a polynomial of any degree. We then connected Horner's method of synthetic division to the Remainder Theorem and the Zero Product Property. This paper also purposely provided two distinct types of instructional aids for the reader: (A) an applet for student experimentation and observation of results and (B) sections of student investigations to accompany major topics. Altogether, we hope that the style and technique of this paper engages the reader and invites the reader to consider both synthetic division and other mathematical topics more deeply.

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Appendix

This appendix contains problems for students to investigate regarding Synthetic Division, Horner's Form, and the Remainder Theorem. These questions are excellent for classroom discussions or to gain a deeper understanding of this mathematics in this article. In order to investigate ideas associated with polynomial long division and synthetic division, we have developed and provided an online applet (https://billcookmath.com/sage/algebra/Horners_method.html). Enjoy.

Student Investigations on Synthetic Division

1. Create your own real polynomial $P(x)$ of degree greater than or equal to 5. Perform synthetic division on $P(x)$ by each of the following: $x-2$; $x+3$; $x-\frac{1}{2}$; $x-\sqrt{2}$; $x-(2-3i)$; $2x-3$; and $\frac{1}{2}x+2$. Check each result using the applet.

2. Reexamine the looping structures in Figure 2. Define the looping structure for the example of the extended polynomial division $\dfrac{x^4+x^2-x+5}{x^2+2x+3}$ provided above.

3. Create your own real polynomial $Q(x)$ with degree of 2 or 3. Perform the extended synthetic division $P(x) \div Q(x)$. Check your result using the applet.

4. Repeat the previous problem with a new polynomial $Q(x)$. Check your result using the applet.

5. Experiment with the applet (https://billcookmath.com/sage/algebra/Horners_method.html) and see what else you can do in respect to polynomial and synthetic division.

Student Investigations Regarding Horner's Form

1. Create your own cubic polynomial $P(x)$. Evaluate $P(3)$. Factor $P(x)$ into Horner's Form and evaluate $P(3)$. Then perform the synthetic division $P(x) \div (x-3)$. Compare all of your result and compare the amount of work which was necessary for each.

2. Create your own real seventh degree polynomial $P(x)$. Evaluate $P(3)$. Factor $P(x)$ into Horner's Form and evaluate $P(3)$. Then perform the synthetic division $P(x) \div (x-3)$. Compare all of your result and compare the amount of work which was necessary for each.

3. In your own words, describe every step in the process of Horner's Method of synthetic division of $\dfrac{n_3x^3+n_2x^2+n_1x+n_0}{d_2x^2+d_1x+d_0}$. (Hint. You may wish to return to your answer for problem 3 in the previous section.)

4. Analyze all of the previous ideas in this paper. Synthesize these ideas into new mathematical connections that have not yet been presented.

Student Investigations Regarding the Remainder Theorem

1. We previously defined the Remainder Theorem as: $\dfrac{P(x)}{x-c} = Q(x) + \dfrac{R}{x-c}$ OR $P(x) = Q(x)(x-c) + R$. Explain this theorem in your own words through an example of polynomials which you have worked out. Then, explain the theorem again limiting your explanation to using $P(x)$, $Q(x)$, $(x-c)$, and R.

2. The Zero Product Property states that for all Complex numbers (including Reals), if $a \cdot b = 0$, then $a = 0$ or $b = 0$. This theorem has been sneakily at work throughout this entire paper. Explain why this theorem is important in respect to the Remainder Theorem.

3. The Zero Product Property from the previous question tells us very much about the possible values of a and b. However, if $a \cdot b = 3$, what do we know about the possible values of a and b?

4. The footnote in the section regarding the Remainder Theorem contains much mathematical notation and symbolism. Define/explain each of the following:

   (a) $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$;

   (b) unique, polynomial, field, root of f;

   (c) $k[x] = \{p(x) \mid p$ is a polynomial in x with coefficients in $k\}$;

   (d) $\deg(r) < \deg(d)$.

5. Create three examples of irreducible quadratic polynomials with integer coefficients.

6. Select some integral values for a, b, and $k > 0$. (Note that only k needs to be greater than zero.). Expand $(ax+b)^2 + k$. Determine the roots of this quadratic. Change the values of a, b, and k, expand the quadratic, and determine its roots. Do this one more time. What do you notice about the roots of these quadratics?

7. Choose any values for a, b, and c such that $b^2 < 4ac$. Write these in the form $ax^2 + bx + c$. Determine the roots of this quadratic. Change the values of a, b, and c (maintaining the relationship $b^2 < 4ac$) and determine the roots of this quadratic. Do this one more time. What do you notice about the roots of these quadratics?

8. Extend the analysis above of substituting the roots of a quadratic in $P(r_i)$ and relating to $R(r_i)$ to any degree divisor.

9. Let us tinker for a moment with the Remainder Theorem. If we begin with $P(x) = Q(x)(x-c) + R$, we can then write $P(x) = Q(x)(x-c) + P(c)$. With simple algebra, we can rearrange this to
$$\frac{P(x) - P(c)}{x-c} = Q(x).$$
In this context, many students have seen $\dfrac{P(x) - P(c)}{x-c}$ as the limit of $P(x)$ as x approaches c and $Q(c)$ as the slope of the line tangent to $P(x)$ at $x = c$. Explain how these ideas are connected.