An Infinite Talk

S-STEM Seminar
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Mathematical Sciences

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Outline

Infinity and Limits

What is ∞?

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Consider $\lim_{x \to \infty} \frac{1}{x}$.

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$\frac{1}{\text{big}}$ = small so $\lim_{x \to \infty} \frac{1}{x} = 0$.

Infinity and Limits

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$\frac{1}{\text{big}}$ = small so $\lim_{x\to\infty} \frac{1}{x} = 0$.

In fact, we could say $\frac{1}{\infty} = 0$.

Infinity and Limits

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Infinity and Limits

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So $\lim_{x\to 0} \dfrac{1}{x}$ does not exist, but $\lim_{x\to 0^+} \dfrac{1}{x} = \infty$ and $\lim_{x\to 0^-} \dfrac{1}{x} = -\infty$.

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In fact, we might say $\dfrac{1}{0^+} = +\infty$ and $\dfrac{1}{0^-} = -\infty$.

Infinity and Limits

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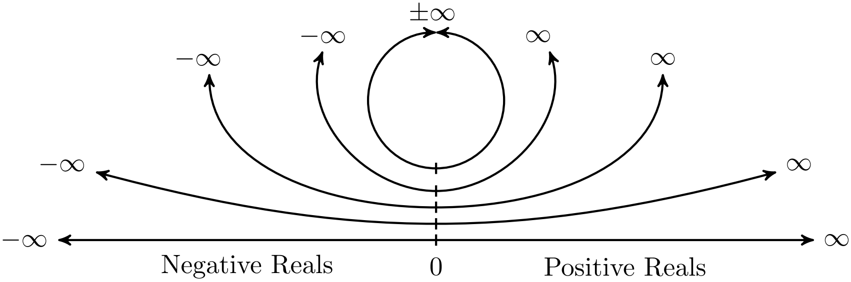
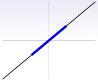
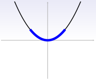
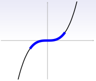
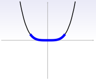
[More precisely, 0^-.]

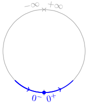
Compactifying the Reals

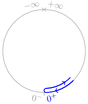


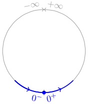
One point compactification of the real numbers = A Circle

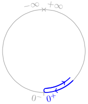





$y = x^1 = x$ $\quad$ $y = x^2$ $\quad$ $y = x^3$ $\quad$ $y = x^4$

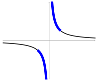
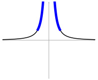
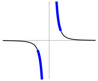
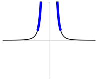




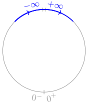


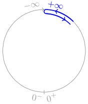


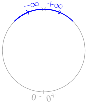
$\Uparrow$ Roots vs. Asymptotes $\Downarrow$

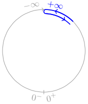





$y = x^{-1} = \frac{1}{x}$ $\quad$ $y = x^{-2} = \frac{1}{x^2}$ $\quad$ $y = x^{-3} = \frac{1}{x^3}$ $\quad$ $y = x^{-4} = \frac{1}{x^4}$

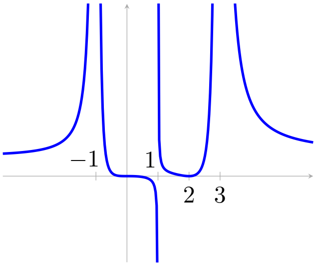








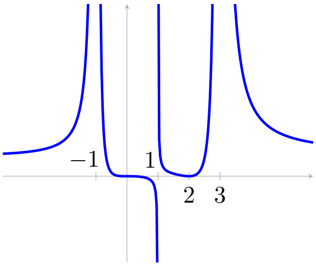
Back to Precalculus



$$f(x) = \frac{2x^3(x-2)^2}{(x+1)^2(x-1)(x-3)^2}$$

For $x \approx 3$, $f(x) \approx \dfrac{2 \cdot 3^3 \cdot 1^2}{4^2 \cdot 2 \cdot (0^{\pm})^2} = \dfrac{54}{32 \cdot 0^+} = \dfrac{54}{0^+} = 54(+\infty) = +\infty$, so "$f(3) = +\infty$".

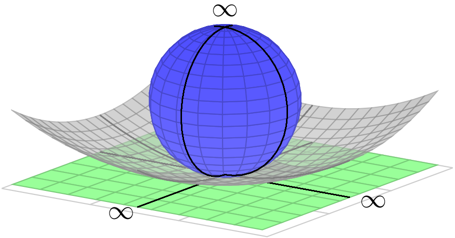
Back to Precalculus



$$f(x) = \frac{2x^3(x-2)^2}{(x+1)^2(x-1)(x-3)^2}$$

For large x, $f(x) \approx \dfrac{2x^3x^2}{x^2xx^2} = \dfrac{2x^5}{x^5} = 2$, so "$f(\pm\infty) = 2$".

Compactifying the Complex Numbers



One point compactification of the complex numbers = The Riemann Sphere

How big is our set?

- $\{1, 2, 3\}$

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- $\{1, 2, 3\}$, $\{999999, -12, 0\}$

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1	$\longleftrightarrow$	999999	$\longleftrightarrow$	a	$\longleftrightarrow$	☺
2	$\longleftrightarrow$	-12	$\longleftrightarrow$	b	$\longleftrightarrow$	😐
3	$\longleftrightarrow$	0	$\longleftrightarrow$	c	$\longleftrightarrow$	☹

Cardinality 3: Sets with 3 elements.

- The empty set, $\emptyset = \{\}$, is the *only set* with 0 elements.

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$$\begin{array}{ccccccc} 1 & \longleftrightarrow & 999999 & \longleftrightarrow & a & \longleftrightarrow & ☺ \\ 2 & \longleftrightarrow & -12 & \longleftrightarrow & b & \longleftrightarrow & 😐 \\ 3 & \longleftrightarrow & 0 & \longleftrightarrow & c & \longleftrightarrow & ☹ \end{array}$$

Cardinality 3: Sets with 3 elements.

- The empty set, $\emptyset = \{\}$, is the *only set* with 0 elements.
- What about $\mathbb{N} = \{0,1,2,\dots\}$?

How big is our set?

We say that sets X and Y have the same *cardinality* if there is an invertible function between them.

Typically we write something like $\mathrm{card}(X) = \mathrm{card}(Y)$ or $|X| = |Y|$ to denote this.

For example: $0 \leftrightarrow 0$, $-1 \leftrightarrow 1$, $1 \leftrightarrow 2$, $-2 \leftrightarrow 3$, $2 \leftrightarrow 4$, $\ldots$
Specifically,

$$f : \mathbb{Z} \to \mathbb{N} \quad \text{defined by} \quad f(x) = \begin{cases} 2x & x \geq 0 \\ -2x - 1 & x < 0 \end{cases}$$

shows that $|\mathbb{Z}| = |\mathbb{N}|$.

Infinite and Countable

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Essentially, countable sets (countably infinite or finite) are the sets we can list off.

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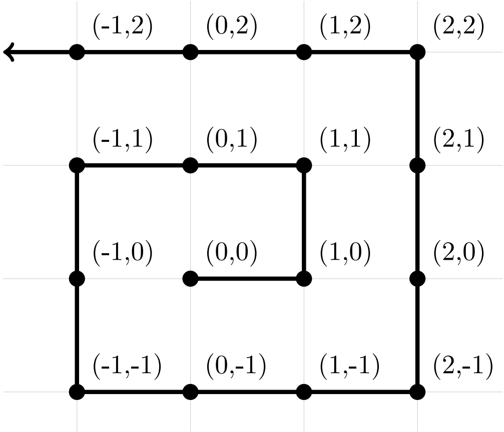
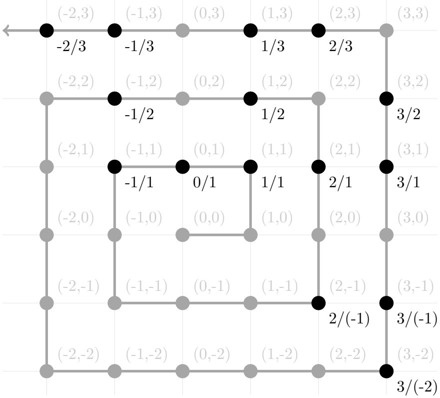


$\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ is countable.

Infinite and Countable



$\mathbb{Q}$ is countable.

Uncountable?

In the world of cardinality, countable infinity (i.e., $\aleph_0$) is the *smallest* kind of infinity.

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Smallest? We say $|X| \le |Y|$ if there is a function from Y to X whose range is all of X.[1]

Also, $|X| < |Y|$ if $|X| \le |Y|$ but $|X| \ne |Y|$.

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Example: $\mathcal{P}(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

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Cantor showed that $|X| < |\mathcal{P}(X)|$.

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We should have some a in X such that $f(a) = \text{Trouble}$.

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$$\text{Trouble} = \{x \in X \mid x \text{ does not belong to } f(x)\}$$

We should have some a in X such that $f(a) = \text{Trouble}$.However, notice that this cannot be since...

a belongs to Trouble exactly when a does not belong to $f(a) =$Trouble!!!

Uncountable?

Theorem: $|X| < |\mathcal{P}(X)|$.

Proof: This is easily seen to be true for the empty set. Let X be non-empty. First, notice that $|X| \leq |\mathcal{P}(X)|$ since we could send $\{x\}$ to x and all other sets to some particular x_0 in X.
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Thus there is no such function f and we have that $|X| \neq |\mathcal{P}(X)|$.

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Corollary: There are infinitely many infinite cardinalities!

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \cdots$$

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Notice that if $A \in \mathcal{U}$, then...

$$A \text{ belongs to } A \text{ exactly when } A \text{ does not belong to } A!$$

Therefore, A cannot belong to our universe $\mathcal{U}$.

The Real Numbers are Uncountable

Cantor's Diagonalization Argument: If $\mathbb{R}$ were countable, we could list off all the real numbers. In particular, we could list all numbers x such that $0 \leq x < 1$, say

$$x_1, x_2, x_3, \ldots$$

Now every real number has a unique decimal expansion if we do not allow trailing 9's like $0.123\overline{9}$ $(= 0.124)$. Write out such expansions:

$$\begin{aligned} x_1 &= 0.\circled{d_{11}} d_{12}\ d_{13}\ d_{14} \ldots \\ x_2 &= 0.\ d_{21} \circled{d_{22}} d_{23}\ d_{24} \ldots \\ x_3 &= 0.\ d_{31}\ d_{32} \circled{d_{33}} d_{34} \ldots \\ &\vdots \end{aligned}$$

Define y_i to be 1 if $y_i = 0$ and $y_i = d_{ii} - 1$ otherwise. Then $\boxed{y = 0.y_1 y_2 y_3 \ldots}$ is a real number between 0 and 1 that's missing from our list!

Cardinal Arithmetic

- $2 + 3 = |\{a, b\}| + |\{a, b, c\}| = |\{a, b\} \dot{\cup} \{a, b, c\}| = |\{\underline{a}, \underline{b}, a, b, c\}| = 5$

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Theorem: Let X and Y be non-empty sets and at least one is infinite. Then...

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Just kidding. This is the *continuum hypothesis*. Gödel showed it cannot be disproven in standard (ZFC) set theory. Cohen proved it cannot be proven either!

Ordinal vs. Cardinal

We often use ordinals and cardinals interchangeably:

$$1,\quad 2,\quad 3 \qquad\qquad \text{vs.} \qquad\qquad 1^{\text{st}},\quad 2^{\text{nd}},\quad 3^{\text{rd}}$$

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This is no longer the case for infinite sets!

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We could order the natural numbers in many ways:

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The axiom of choice says that every set can be well ordered. Homework: Well order $\mathbb{R}$.

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Ordinals are well-ordered by set containment.

Building Ordinals

- $0 = \emptyset \qquad 1 = \{0\} \qquad 2 = \{0, 1\}$
- $n + 1 = n \cup \{n\}$
- $\omega = \bigcup_{n=0}^{\infty} n = \{0, 1, 2, \dots\}$
- $\omega + 1 = \{0, 1, 2, \dots, \omega\}$
- $\omega + 2 = \{0, 1, 2, \dots, \omega, \omega + 1\}$
- $\omega + \omega = \{0, 1, 2, \dots, \omega, \omega + 1, \omega + 2, \dots\}$

In general, we define the successor of an ordinal number κ to be $\kappa + 1 = \kappa \cup \{\kappa\}$.
We also let the (arbitrary union) of an ordinal number be an ordinal number.

Ordinals are well-ordered by set containment.Essentially, the axiom of choice says:
Given any set, there is an invertible function between that set and a (unique) ordinal number.

Ordinal Arithmetic is Weirder

We can add ordinals (think of tacking on onto the end of the other) as in our examples:

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- $\omega + 2 = \{0, 1, 2, \ldots, \omega, \omega + 1\}$
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- $2 + \omega = \{0, 1\} + \{0, 1, 2, \ldots\} = \{\underline{0}, \underline{1}, 0, 1, 2, \ldots\} = \omega$ so $\omega + 2 \neq 2 + \omega$

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  So $\omega \cdot 2 = \omega + \omega$ $(\neq 2 \cdot \omega)$.
- $\omega \cdot \omega$ is like listing the natural numbers over and over again (countably) infinitely many times.

Ordinal Arithmetic is Weirder

We could also define ordinal exponentiation (but we won't).

Another strange calculation:

$$(\omega \cdot 2)^2 = (\omega \cdot 2) \cdot (\omega \cdot 2) = \omega \cdot (2 \cdot \omega) \cdot 2 = \omega \cdot \omega \cdot 2 = \omega^2 \cdot 2 = \omega^2 + \omega^2$$

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As one last bizarre example, stated without explanation or proof:

$$2^\omega = \omega$$

Ordinal arithmetic can be very strange.

Cardinals as Ordinals

A cardinal number can be realized as an ordinal number.

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- $|\mathbb{N}| = \aleph_0 = \omega = \mathbb{N} = \{0, 1, 2, \ldots\}$
- $|\omega + \omega| = \aleph_0$

Questions?!?

Further reading / more details:

- "Crossing Through and Bouncing Off ∞: Graphing Rational Functions" with Michael Bossé published in MathAMATYC Educator, Winter Issue 2021, Vol. 12 Number 1.
- "Types of Infinity" with Michael Bossé published in The Electronic Journal of Mathematics & Technology, Volume 15 (2021) Number 2 (June).

Both papers are available on my website:

http://BillCookMath.com/papers