FUCHS' PROBLEM FOR SMALL GROUPS

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Abstract. Fuchs' problem asks which groups can be realized as unit groups. In this paper, we solve Fuchs' problem for dicyclic groups realized by finite rings. We also survey known results and give complete lists of realizable groups considering groups up to order 15. For groups that can be realized, we provide a ring in every viable characteristic. Consequently, we have that the dicyclic group of order 12 is the smallest group that can be realized but not by a finite ring.

Keywords: Fuchs' problem, group of units, dicyclic groups
MSC 2010: 16U60 (Primary), 20C05 (Secondary)

1. Introduction and General Background

Over 60 years ago, László Fuchs asked which Abelian groups can appear as groups of units of a ring ([6]). Fuchs' problem now refers to the classification of which groups (Abelian or not) are unit groups of rings. While this problem has been resolved for many classes of groups, for example cyclic groups ([8] and [9]), dihedral groups ([2]), alternating and symmetric groups ([3]), and finite simple groups ([4]), Fuchs' original question about Abelian groups is still open.

In this paper, we answer Fuchs' question for groups up to order 15. In addition, we explore the class of dicyclic groups where we prove that dicyclic groups of order 12 or larger cannot be groups of units of a finite ring. We also give partial results for and show certain dicyclic groups can be realized in characteristic 0 while others cannot. An interesting consequence of this work is that the dicyclic group of order 12 is the smallest group to appear as a group of units of an infinite ring but not as the group of units of a finite ring.

To begin, we must establish some terminology and notation. All of our rings possess a multiplicative identity and we let $R^\times$ denote the group of units of a ring R. For convenience, we occasionally let $\mathbb{Z}_0$ denote the integers, $\mathbb{Z}$, and we identify the prime subring of a ring of characteristic n with $\mathbb{Z}_n$. In addition, $\mathbb{F}_q$ denotes the finite field of order q, C_n denotes the cyclic group of order n, D_{2n} denotes the dihedral group of order $2n$, and Dic_{4n} denotes the dicyclic group of order $4n$.

Definition 1.1. *A group is* **realizable** *if it is the group of units of some ring. Moreover, a group is* **realizable in characteristic** n *if it is the group of units of a ring of characteristic* n.

Let G be a group and R a commutative ring. Recall that

$$R[G] = \left\{ \sum_{i=1}^{n} r_i g_i \;\middle|\; n \in \mathbb{Z}_{\geq 0} \text{ and for all } i = 1, \ldots, n \text{ we have } r_i \in R \text{ and } g_i \in G \right\}$$

is the group ring of G with coefficients in R. These elements are added coefficientwise. If $r, s \in R$ and $g, h \in G$, we have $(rg)(sh) = rs \cdot gh$ and then extend linearly for general elements of $R[G]$. Notice that $G \subseteq (R[G])^\times$. Therefore, any group appears *among* the units of some ring of arbitrary characteristic. Fuchs' problem is much more difficult. He asks if we can find a ring R such that $G = R^\times$ precisely. That is, we have equality and not just containment.

Let us gather some basic facts.

Proposition 1.2. *Let G be a group and n a non-negative integer. Suppose that G is realized by a ring R of characteristic n.*

(a) There is an ideal I of $\mathbb{Z}_n[G]$ such that $(\mathbb{Z}_n[G]/I)^\times \cong G$. Moreover, G can be realized by a finite ring if and only if G is finite and realizable in a positive characteristic.
(b) Let S be the subring of R generated by the units G. Then $S^\times = G$.
(c) If a ring realizes an Abelian group, one of its commutative subrings realizes that group.
(d) If G is cyclic, then there is some ideal I of $\mathbb{Z}_n[x]$ such that $(\mathbb{Z}_n[x]/I)^\times \cong G$.

(e) *Units of the prime subring are central units:* $\mathbb{Z}_n^\times \subseteq Z(R^\times) = Z(G)$.
(f) *We have* -1 *is a unit of order 1 or 2. Moreover,* -1 *is a unit of order 2 if and only if the characteristic of* R *is not 1 or 2.*
(g) *Non-trivial realizable groups of odd order must be realized in characteristic 2.*

Proof: Suppose $R^\times = G$ where R is a ring of characteristic n. The identity map on G extends to a ring homomorphism $\varphi : \mathbb{Z}_n[G] \to R$. Thus by the first isomorphism theorem $\mathbb{Z}_n[G]/\ker(\varphi) \cong \mathrm{im}(\varphi) \subseteq R$. Notice that $S = \mathrm{im}(\varphi)$ is precisely the subring of R generated by the unit group G. Moreover, since inverses of units are units and S contains all of $G = R^\times$, we have that $S^\times = R^\times$.

Next, if $G = R^\times$ is infinite (so $G \subseteq R$) or the characteristic of our ring is $n = 0$ (so $\mathbb{Z} \subseteq R$), then R must be an infinite ring. On the other hand, if G is finite and can be realized in positive characteristic n, then it can be realized by $\mathbb{Z}_n[G]/I$ for some ideal I. This ring is finite since $\mathbb{Z}_n[G]$ is finite. This establishes parts (a) and (b). Part (c) now follows from part (b) since a set of commuting elements generates a commutative subring.

Part (d) is much like part (a). Suppose $G = \langle g \rangle$. Then one can extend $x \mapsto g$ to a homomorphism, $\varphi : \mathbb{Z}_n[x] \to R$. The result now follows from the isomorphism theorem.

For part (e), we note that units of the prime subring $\mathbb{Z}_n$ are units of R, and the prime subring $\mathbb{Z}_n$ is central (i.e., $\mathbb{Z}_n \subseteq Z(R)$). Thus $\mathbb{Z}_n^\times \subseteq Z(R^\times) = Z(G)$.

For part (f), we note that $(-1)^2 = 1$, so -1 is a unit of order 1 or 2. Notice that -1 has order 1 if and only if $-1 = 1$ (i.e., $2 = 0$). This is the case if and only if the characteristic is either 1 or 2. Finally, part (g) follows immediately from (f) since a group of odd order cannot possess an element of even order. □

We say that A is a maximal Abelian subgroup of a group G if A is an Abelian subgroup of G and given any other Abelian subgroup B of G such that $A \subseteq B$, we have $A = B$. In particular, we do not require A to be a proper subgroup. We record the following helpful fact:

Proposition 1.3. *Let* R *be a ring and let* A *be a maximal Abelian subgroup of* $R^\times$. *Then* A *can be realized by a subring of* R. *Therefore, if* $R^\times$ *is realizable in characteristic* n, *then every maximal Abelian subgroup of* $R^\times$ *is also realizable in characteristic* n.

Proof: Let A be a maximal Abelian subgroup of $R^\times$ where $\mathrm{char}(R) = n$. Consider the subring of R generated by A, namely $S = \mathrm{span}_{\mathbb{Z}_n}(A)$ (where if $\mathrm{char}(R) = 0$, then $\mathbb{Z}_n = \mathbb{Z}$). Then S is commutative, $S^\times$ is Abelian, and $A \subseteq S^\times \subseteq R^\times$. Since A is a maximal Abelian subgroup, $A = S^\times$. □

While the above propositions help us limit how groups can be realized, the following construction is helps us build some realizations:

Remark 1.4. *Recall that if we have two rings* R *and* S *along with the ring homomorphisms* $f, g : S \to Z(R)$ *where* $Z(R)$ *is the center of* R, *then we can construct a semidirect product ring* $R \rtimes S$ *with underlying set* $R \times S$. *The additive structure is just that of the direct product of* R *and* S, *and we multiply as follows:* $(a, b)(x, y) = (af(y) + xg(b), by)$ *for* $a, x \in R$ *and* $b, y \in S$. *It is a routine exercise to show that this gives* $R \rtimes S$ *a ring structure with unit* $1 = (0, 1)$. *Moreover,* (x, y) *is a unit of* $R \rtimes S$ *if and only if* y *is a unit of* S. *In fact,* $(x, y)^{-1} = (-xg(y^{-1})f(y^{-1}), y^{-1})$. *Thus* $(R \rtimes S)^\times = R \times (S^\times) = \{(x, y) \mid x \in R \text{ and } y \in S^\times\}$ *is the set of units of our semidirect product.*

2. Groups of Odd Order

The fact that -1 is a unit of order 2 greatly limits our options for realizing groups of odd order. In particular, the final part of Proposition 1.2 states that a non-trivial group of odd order cannot be realized except in characteristic 2. This implies that an odd order group can be realized if and only if it can be realized by a finite ring (in particular, a quotient of its group ring with coefficients in $\mathbb{Z}_2$).

Let us dismiss the trivial group. Note that the trivial ring is the only ring of characteristic 1, and so the trivial group is the only group realizable in characteristic 1. We can also realize the trivial group in characteristic 2 using $\mathbb{Z}_2$ (the only other characteristic available). In fact, the smallest rings to realize the trivial group in characteristics 1 and 2 are $\{0\}$ and $\mathbb{Z}_2$ $(= \mathbb{F}_2)$ respectively.

All odd ordered groups of orders 15 or less are Abelian, and other than $\mathbb{Z}_3 \times \mathbb{Z}_3$ they are cyclic. Since [8] solved Fuchs' problem for cyclic groups realized by finite rings, our groups of odd order, except $\mathbb{Z}_3 \times \mathbb{Z}_3$, could be handled by his result. Instead, we provide a direct proof to demonstrate some tools at hand.

Proposition 2.1. *The cyclic groups C_5, C_9, C_{11}, and C_{13} cannot be realized.*

Proof: Let $C_n = \langle s \rangle$ be a cyclic group of order $n > 1$. Suppose that we can realize C_n by a ring R. By Proposition 1.2 parts (g) and (d), we may assume that we have a surjective homomorphism $\varphi : \mathbb{Z}_2[x] \to R$ such that $\varphi(x) = s$. Since $s^n = 1$ (so $0 = s^n - 1 = s^n + 1$ in characteristic 2), we have that $\varphi(x^n + 1) = 0$. Therefore, this morphism can be factored through $\mathbb{Z}_2[x]/(x^n + 1)$. In other words, R is isomorphic to a quotient of $\mathbb{Z}_2[x]/(x^n + 1)$.

When $n = 5$, 11, or 13, the polynomial x^n+1 factors in $\mathbb{Z}_2[x]$ as follows: $x^n+1 = (x+1)(x^{n-1}+\cdots+x^2+x+1)$. Therefore, R is isomorphic to a quotient of $\mathbb{Z}_2[x]/(x^n+1) = \mathbb{Z}_2[x]/(x+1)\times\mathbb{Z}_2[x]/(x^{n-1}+\cdots+x+1) \cong \mathbb{F}_{2^1} \times \mathbb{F}_{2^{n-1}}$. Recalling that ideals of product rings are products of ideals and that fields have no non-trivial proper quotients, we must have that R is isomorphic with either $\{0\}$, $\mathbb{F}_2$, $\mathbb{F}_{2^{n-1}}$, or $\mathbb{F}_{2^1} \times \mathbb{F}_{2^{n-1}}$. These yield possible unit groups of C_1, C_1, $C_{2^{n-1}-1}$, and $C_{2^{n-1}-1}$ respectively. But 1 and $2^{n-1} - 1$ do not equal n when $n = 5, 11$, or 13. Thus C_5, C_{11}, and C_{13} cannot be realized.

Using the same line of reasoning, we factor x^9+1 in $\mathbb{Z}_2[x]$ as follows: $x^9+1 = (x+1)(x^2+x+1)(x^6+x^3+1)$. Therefore, if C_9 can be realized, our ring must be isomorphic to a quotient of $\mathbb{F}_2 \times \mathbb{F}_{2^2} \times \mathbb{F}_{2^6}$. So we would need to be able to express C_9 as a product built out of the groups C_1, C_3, and C_{63}. This is impossible, so C_9 cannot be realized. □

The rest of the groups of odd order (up to order 15) can be realized as unit groups of finite fields (or products of such). The following table gives the smallest possible realization in all possible characteristics c for each group of odd order less than or equal to 15:

C_1	C_3	C_5	C_7	C_9	$C_3 \times C_3$	C_{11}	C_{13}	C_{15}
$c = 1$: $\{0\}$ $c = 2$: $\mathbb{F}_2$	$c = 2$: $\mathbb{F}_4$	None	$c = 2$: $\mathbb{F}_8$	None	$c = 2$: $\mathbb{F}_4 \times \mathbb{F}_4$	None	None	$c = 2$: $\mathbb{F}_{16}$

3. Cyclic Groups of Even Order

In this section we consider cyclic groups of even order less than 15. We recall that Fuchs' problem is completely solved for all cyclic groups. In particular, Gilmer showed which cyclic groups can be realized by a finite ring and then Pearson and Schneider took care of the remaining characteristic 0 case ([8] and [9]). These results are recovered and extended in [5] (we quote their Corollary 5.17):

Theorem 3.1. ([9]) *A finite cyclic group is realizable if its order is the product of a set of pairwise relatively prime integers drawn from the following list:*
(a) $p^k - 1$ where p is prime and $k \geq 1$;
(b) $(p - 1)p^k$ where $p > 2$ and $k \geq 1$;
(c) $2k$ where k is odd;
(d) $4k$ where any prime dividing k is congruent to 1 modulo 4.

We will not need this theorem here, but we do need Gilmer's main result. Instead of reproving it, we merely recall Gilmer's theorem. This will allow us to determine which cyclic groups of even order can be realized in characteristic $n \neq 0$. Most of these cyclic groups of even order can be realized in characteristic 0 with the exception of C_8, so instead of relying on Pearson and Schneider's result, we will provide realizations and in the case of C_8 we provide a direct proof that it cannot be realized in characteristic 0.

Theorem 3.2. ([8]) *(I) Let R be a finite commutative ring. Then $R = R_1 \times R_2 \times \cdots \times R_m$ where $R_1, \ldots, R_m$ are primary rings. Moreover, $R^\times$ is cyclic if and only if $R_i^\times$ is cyclic for each $i = 1, \ldots, m$ and the orders of $R_i^\times$ $(i = 1, \ldots, m)$ are relatively prime.*

(II) Let R be a (non-trivial) finite primary ring with cyclic group of units. Then R is isomorphic to one of the following:[1]

- $\mathbb{F}_{p^k}$ *where p is prime and $k > 0$. Note:* $\mathrm{char}(\mathbb{F}_{p^k}) = p$ *and* $(\mathbb{F}_{p^k})^\times \cong C_{p^k-1}$.

[1] Appendix B includes a verification that $\mathbb{Z}_p^\varepsilon$, $\mathbb{Z}_2^\delta$, and Gil do in fact have these unit groups.

- $\mathbb{Z}_{p^k}$ *where* p *is an odd prime and* $k > 1$ *or* $p^k = 2^2 = 4$. *Note:* $\text{char}(\mathbb{Z}_{p^k}) = p^k$ *and* $(\mathbb{Z}_{p^k})^\times \cong C_{p^k - p^{k-1}}$.
- $\mathbb{Z}_p^\varepsilon = \mathbb{Z}_p[x]/(x^2)$ *where* p *is prime. Note:* $\text{char}(\mathbb{Z}_p^\varepsilon) = p$ *and* $(\mathbb{Z}_p^\varepsilon)^\times \cong C_{p^2-p}$.
- $\mathbb{Z}_2^\delta = \mathbb{Z}_2[x]/(x^3)$. *Note:* $\text{char}(\mathbb{Z}_2^\delta) = 2$ *and* $(\mathbb{Z}_2^\delta)^\times \cong C_4$.
- $\text{Gil} = \mathbb{Z}_4[x]/(2x, x^2+2)$. *Note:* $\text{char}(\text{Gil}) = 4$ *and* $\text{Gil}^\times \cong C_4$.

(III) If C_n *is realizable, it can be realized by a finite commutative ring and thus can be realized by a direct product of rings drawn from the above list.*

With Gilmer's theorem in hand, we need to consider which direct products of the rings above give us a cyclic group of order 15 or less. First, finite fields give us $(\mathbb{Z}_2)^\times \cong C_1$, $(\mathbb{F}_4)^\times \cong C_3$, $(\mathbb{F}_8)^\times \cong C_7$, $(\mathbb{F}_{16})^\times \cong C_{15}$, $(\mathbb{Z}_3)^\times \cong C_2$, $(\mathbb{F}_9)^\times \cong C_8$, $(\mathbb{Z}_5)^\times \cong C_4$, $(\mathbb{Z}_7)^\times \cong C_6$, $(\mathbb{Z}_{11})^\times \cong C_{10}$, and $(\mathbb{Z}_{13})^\times \cong C_{12}$. Next, integers modulo a prime power order give us $(\mathbb{Z}_4)^\times \cong C_2$ and $(\mathbb{Z}_9)^\times \cong C_6$. Next, we have $(\mathbb{Z}_2^\varepsilon)^\times \cong C_2$ and $(\mathbb{Z}_3^\varepsilon)^\times \cong C_6$. Finally, $(\mathbb{Z}_2^\delta)^\times \cong C_4$ and $\text{Gil}^\times \cong C_4$. Everything else yields cyclic groups with more than 15 elements.

With our positive characteristic building blocks in place, we then turn to characteristic 0. In [9], Pearson and Schneider extend Gilmer's result covering all characteristics. We already know that groups of odd order cannot be realized in characteristic 0, and it turns out that the only other cyclic group of order less than 15 that cannot be realized in characteristic 0 is C_8. Instead of quoting Pearson and Schneider's results, we offer a direct proof to show C_8 cannot be realized in characteristic 0.

Lemma 3.3. *Suppose a ring* R *realizes* $R^\times = C_8$. *Then* $\text{char}(R)$ *must divide 48. In particular,* $\text{char}(R) \neq 0$.

Proof: Suppose $\text{char}(R)$ does not divide 48. Thus $\text{char}(R) \neq 2$ and so -1 must be our unique unit of order 2. Let $\omega \in R^\times$ be some generator of $R^\times = C_8$. Then $\omega^4 = -1$ since ω^4 has order 2. Next, notice that $(\omega^2+\omega+1)(\omega^6+\omega^3+1) = \omega^8+\omega^7+\cdots+1 = 1-\omega^3-\omega^2-\omega-1+\omega^3+\omega^2+\omega+1 = 1$. Thus $\omega^2+\omega+1$ is a unit in R and so $(\omega^2+\omega+1)^8 = 1$. Running the extended Euclidean algorithm on $(x^2+x+1)^8-1$ and x^4+1, one finds that there exist $f, g \in \mathbb{Z}[x]$ such that $f(x)((x^2+x+1)^8-1)+g(x)(x^4+1) = 48$. Therefore, $48 = f(\omega)((\omega^2+\omega+1)^8-1)+g(\omega)(\omega^4+1) = f(\omega)\cdot 0 + g(\omega)\cdot 0 = 0$. This is impossible since the characteristic of R does not divide 48. □

The other cyclic groups of even orders less than 15 can be realized in characteristic 0 if we form direct products of previously considered rings with the integers, the Gaussian integers, and $PS_5 = \mathbb{Z}[x]/(5x, x^2)$ (i.e., Ring A with $m = 5$ from [9]). In particular, we need PS_5 to realize C_{10} in characteristic 0.[2]

After taking direct products of our list coming from Gilmer with these characteristic 0 rings, we get the following table of realizations of cyclic groups of even orders less than 15 in all viable characteristics:

C_2		C_4		C_6		
$c=0$: $\mathbb{Z}$	$c=4$: $\mathbb{Z}_4$	$c=0$: $\mathbb{Z}[i]$	$c=5$: $\mathbb{Z}_5$	$c=0$: $\mathbb{F}_4 \times \mathbb{Z}$	$c=4$: $\mathbb{Z}_4 \times \mathbb{F}_4$	$c=9$: $\mathbb{Z}_9$
$c=2$: $\mathbb{Z}_2^\varepsilon$	$c=6$: $\mathbb{Z}_6$	$c=2$: $\mathbb{Z}_2^\delta$	$c=10$: $\mathbb{Z}_{10}$	$c=2$: $\mathbb{Z}_2^\varepsilon \times \mathbb{F}_4$	$c=6$: $\mathbb{Z}_3 \times \mathbb{F}_4$	$c=14$: $\mathbb{Z}_{14}$
$c=3$: $\mathbb{Z}_3$		$c=4$: Gil		$c=3$: $\mathbb{Z}_3^\varepsilon$	$c=7$: $\mathbb{Z}_7$	$c=18$: $\mathbb{Z}_{18}$

C_8	C_{10}	C_{12}		C_{14}	
$c=3$: $\mathbb{F}_9$	$c=0$: PS_5	$c=0$: $\mathbb{F}_4 \times \mathbb{Z}[i]$	$c=10$: $\mathbb{F}_4 \times \mathbb{Z}_5$	$c=0$: $\mathbb{F}_8 \times \mathbb{Z}$	$c=4$: $\mathbb{Z}_4 \times \mathbb{F}_8$
$c=6$: $\mathbb{Z}_2 \times \mathbb{F}_9$	$c=11$: $\mathbb{Z}_{11}$	$c=2$: $\mathbb{F}_4 \times \mathbb{Z}_2^\delta$	$c=13$: $\mathbb{Z}_{13}$	$c=2$: $\mathbb{Z}_2^\varepsilon \times \mathbb{F}_8$	$c=6$: $\mathbb{Z}_3 \times \mathbb{F}_8$
	$c=22$: $\mathbb{Z}_{22}$	$c=4$: $\mathbb{F}_4 \times \text{Gil}$	$c=26$: $\mathbb{Z}_{26}$		

4. Non-Cyclic Abelian Groups of Even Orders

There are four non-cyclic Abelian groups of even order less than 15. Each of these can be realized, so our only work is to determine which characteristics are viable. In each case we recall that if $G = Z(G)$ is realized by R in characteristic c, then $\mathbb{Z}_c^\times \subseteq G$ (where we continue to use the convention that $\mathbb{Z}_0 = \mathbb{Z}$). From this consideration, we find if $\text{char}(R) = c$, then:

[2] Appendix B has a verification that for any positive odd integer $(PS_m)^\times = (\mathbb{Z}[x]/(mx, x^2))^\times = C_{2m}$.

For $C_2 \times C_2$, $c = 0, 2, 3, 4, 6, 8,$ and 12 may be possible.
For $C_2 \times C_2 \times C_2$, $c = 0, 2, 3, 4, 6, 8, 12,$ and 24 may be possible.
For $C_2 \times C_4$, $c = 0, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20,$ and 30 may be possible.
For $C_2 \times C_6$, $c = 0, 2, 3, 4, 6, 7, 8, 9, 12, 14, 18, 21, 28, 36,$ and 42 may be possible.

As we will see below, nearly every one of these possible characteristics can be realized. Only three cases need to be ruled out. We show that $C_2 \times C_4$ cannot be realized in characteristics 3 and 5, and $C_2 \times C_6$ cannot be realized in characteristic 7.

Lemma 4.1. *Suppose R realizes $C_2 \times C_4$ in characteristic c. Then* $\text{char}(R) \neq 3$.

Proof: Suppose $\text{char}(R) = 3$ and let $C_2 \times C_4 = \langle y \rangle \times \langle x \rangle = \{1, y, x, xy, x^2, x^2y, x^3, x^3y\}$ so that y, x^2, x^2y have order 2 and x, x^3, xy, x^3y have order 4. Note that $2 = -1$ is a unit of order 2. Thus, one of the following must hold: $y = -1$, $x^2 = -1$, or $x^2y = -1$.

If $x^2 = -1$, then $(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4 = 1 + x + 0 + x(-1) + 1 = 2 = -1$ and so $(1+x)^8 = 1$. Thus $1 + x$ is a unit of order 8 which is impossible.

Next, if $y = -1$, then without loss of generality we can assume our ring is a quotient of $\mathbb{Z}_3[x]/(x^4-1)$ (i.e., we send x to x and -1 to y). Note that $x^4 - 1 = (x+1)(x-1)(x^2+1)$ over $\mathbb{Z}_3[x]$. Thus $\mathbb{Z}_3[x]/(x^4-1) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{F}_9$. A quotient of this ring must be isomorphic to a direct product built from $\mathbb{Z}_3$, $\mathbb{Z}_3$, and $\mathbb{F}_9$. Such a quotient can only realize C_1, C_2, $C_2 \times C_2$, $C_2 \times C_8$, C_8, or $C_2 \times C_2 \times C_8$. Therefore, no such quotient ring realizes $C_2 \times C_4$.

Finally, if $x^2y = -1$, we can just let $z = x^2y = -1$ so $C_2 \times C_4 = \{1, x^2z, x, x^3z, x^2, z, x^3, xz\}$ and our previous case applies. Thus neither $x^2 = -1$ nor $y = -1$ nor $x^2y = -1$ is possible. We have reached a contradiction. □

Lemma 4.2. *Suppose R realizes $C_2 \times C_4$ in characteristic c. Then* $\text{char}(R) \neq 5$.

Proof: Let $C_2 \times C_4 = \langle y \rangle \times \langle x \rangle$ be realized by some ring R of characteristic 5. Without loss of generality, assume that R is generated by its units. Our prime subring is $\mathbb{Z}_5$ which contains units $1, 2, 4,$ and 3. These can be identified with $1, x, x^2,$ and x^3 respectively. The remaining units are then $y, 2y, 4y,$ and $3y$. Sending 2 to x and y to y, we get a homomorphism from $\mathbb{Z}_5[y]/(y^2-1)$ to R. This homomorphism is onto since R is generated by its units. Noting that $y^2 - 1 = (y-1)(y+1)$, we have R is isomorphic to a quotient of $\mathbb{Z}_5[y]/(y^2-1) \cong \mathbb{Z}_5 \times \mathbb{Z}_5$. A quotient of this ring must be isomorphic to $\{0\}$, $\mathbb{Z}_5$, or $\mathbb{Z}_5 \times \mathbb{Z}_5$ with unit group C_1, C_4, or $C_4 \times C_4$ respectively. Therefore, no such quotient ring realizes $C_2 \times C_4$. We have reached a contradiction. □

Lemma 4.3. *Suppose R realizes $C_2 \times C_6$ in characteristic c. Then* $\text{char}(R) \neq 7$.

Proof: Let $C_2 \times C_6 = \langle y \rangle \times \langle x \rangle$ be realized by some ring R of characteristic 7. Without loss of generality, assume that R is generated by its units. Our prime subring is $\mathbb{Z}_7$ which contains units $1, 3, 2, 6, 4,$ and 5. These can be identified with $1, x, x^2, x^3, x^4,$ and x^5 respectively. The remaining units are then $y, 3y, 2y, 6y, 4y,$ and $5y$. Sending 3 to x and y to y, we get a homomorphism from $\mathbb{Z}_7[y]/(y^2-1)$ to R. This homomorphism is onto since R is generated by its units. Noting that $y^2 - 1 = (y-1)(y+1)$, we have R is isomorphic to a quotient of $\mathbb{Z}_7[y]/(y^2-1) \cong \mathbb{Z}_7 \times \mathbb{Z}_7$. A quotient of this ring must be isomorphic to $\{0\}$, $\mathbb{Z}_7$, or $\mathbb{Z}_7 \times \mathbb{Z}_7$ with unit group C_1, C_6, or $C_6 \times C_6$ respectively. Therefore, no such quotient ring realizes $C_2 \times C_6$. We have reached a contradiction. □

Recall that $\mathbb{Z}_2^{\varepsilon} = \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2^{\delta} = \mathbb{Z}_2[x]/(x^3)$, $\text{Gil} = \mathbb{Z}_4[x]/(2x, x^2+2)$, and define $\text{M} = \mathbb{Z}_8[x]/(x^2+1, 2x-2)$.[3] Therefore, we get the following table of realizations of non-cyclic Abelian groups of even orders less than 15 in all viable characteristics:

[3] Appendix B includes a verification that $(\mathbb{Z}_2^{\varepsilon})^\times = C_2$, $(\mathbb{Z}_2^{\delta})^\times = C_4$, $(\text{Gil})^\times = C_4$, and $\text{M}^\times = C_2 \times C_4$.

$C_2 \times C_2$	
$c = 0$: $\mathbb{Z} \times \mathbb{Z}$	$c = 6$: $\mathbb{Z}_3 \times \mathbb{Z}_6$
$c = 2$: $\mathbb{Z}_2^\varepsilon \times \mathbb{Z}_2^\varepsilon$	$c = 8$: $\mathbb{Z}_8$
$c = 3$: $\mathbb{Z}_3 \times \mathbb{Z}_3$	$c = 12$: $\mathbb{Z}_{12}$
$c = 4$: $\mathbb{Z}_4 \times \mathbb{Z}_4$	

$C_2 \times C_2 \times C_2$	
$c = 0$: $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$	$c = 6$: $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_6$
$c = 2$: $\mathbb{Z}_2^\varepsilon \times \mathbb{Z}_2^\varepsilon \times \mathbb{Z}_2^\varepsilon$	$c = 8$: $\mathbb{Z}_2^\varepsilon \times \mathbb{Z}_8$
$c = 3$: $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	$c = 12$: $\mathbb{Z}_2^\varepsilon \times \mathbb{Z}_{12}$
$c = 4$: $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$	$c = 24$: $\mathbb{Z}_{24}$

$C_2 \times C_4$	
$c = 0$: $\mathbb{Z} \times \mathbb{Z}[i]$	$c = 12$: $\mathbb{Z}_3 \times$ Gil
$c = 2$: $\mathbb{Z}_2^\varepsilon \times \mathbb{Z}_2^\delta$	$c = 15$: $\mathbb{Z}_{15}$
$c = 4$: $\mathbb{Z}_2^\varepsilon \times$ Gil	$c = 16$: $\mathbb{Z}_{16}$
$c = 6$: $\mathbb{Z}_6 \times \mathbb{Z}_2^\delta$	$c = 20$: $\mathbb{Z}_{20}$
$c = 8$: M	$c = 30$: $\mathbb{Z}_{30}$
$c = 10$: $\mathbb{Z}_2^\varepsilon \times \mathbb{Z}_5$	

$C_2 \times C_6$		
$c = 0$: $\mathbb{Z} \times \mathbb{Z} \times \mathbb{F}_4$	$c = 8$: $\mathbb{Z}_8 \times \mathbb{F}_4$	$c = 21$: $\mathbb{Z}_{21}$
$c = 2$: $\mathbb{Z}_2^\varepsilon \times \mathbb{Z}_2^\varepsilon \times \mathbb{F}_4$	$c = 9$: $\mathbb{Z}_3 \times \mathbb{Z}_9$	$c = 28$: $\mathbb{Z}_{28}$
$c = 3$: $\mathbb{Z}_3 \times \mathbb{Z}_3^\varepsilon$	$c = 12$: $\mathbb{F}_4 \times \mathbb{Z}_{12}$	$c = 36$: $\mathbb{Z}_{36}$
$c = 4$: $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{F}_4$	$c = 14$: $\mathbb{Z}_2^\varepsilon \times \mathbb{Z}_{14}$	$c = 42$: $\mathbb{Z}_{42}$
$c = 6$: $\mathbb{Z}_6 \times \mathbb{Z}_3 \times \mathbb{F}_4$	$c = 18$: $\mathbb{Z}_3 \times \mathbb{Z}_{18}$	

5. Dihedral Groups

For any positive integer n, the dihedral group of order $2n$ can be presented by $D_{2n} = \langle r, s \mid r^n = 1, s^2 = 1, rs = sr^{-1} \rangle$. Of course, $D_2 \cong C_2$ and $D_4 \cong C_2 \times C_2$ (the Klein 4-group). All other dihedral groups are non-Abelian. Chebolu and Lockridge in [2] give a self-contained complete solution to Fuchs' problem. This paper inspired much of our work done in Section 7 on dicyclic groups. Instead of reproving their result, we merely relay their conclusion.

Theorem 5.1. *Suppose R realizes $R^\times = D_{2n}$ in characteristic c. If $n = 1$, then $c = 0, 2, 3, 4,$ or 6. If $n = 2$, then $c = 0, 2, 3, 4, 6, 8,$ or 12. If $n = 3$, then $c = 2$. If $n = 4$, then $c = 2$ or 4. If $n = 6$, then $c = 2, 3, 4,$ or 6. Finally, if $n = 4k$ where k is an odd positive integer, then $c = 0$. All other dihedral groups cannot be realized.*

The following table gives examples of realizations of dihedral groups in the only viable characteristics:

D_2	
$c = 0$: $\mathbb{Z}$	$c = 4$: $\mathbb{Z}_4$
$c = 2$: $\mathbb{Z}_2^\varepsilon$	$c = 6$: $\mathbb{Z}_6$
$c = 3$: $\mathbb{Z}_3$	

D_4	
$c = 0$: $\mathbb{Z} \times \mathbb{Z}$	$c = 6$: $\mathbb{Z}_3 \times \mathbb{Z}_6$
$c = 2$: $\mathbb{Z}_2^\varepsilon \times \mathbb{Z}_2^\varepsilon$	$c = 8$: $\mathbb{Z}_8$
$c = 3$: $\mathbb{Z}_3 \times \mathbb{Z}_3$	$c = 12$: $\mathbb{Z}_{12}$
$c = 4$: $\mathbb{Z}_4 \times \mathbb{Z}_4$	

D_6
$c = 2$: $(\mathbb{Z}_2)^{2\times 2}$

D_8
$c = 2$: $U_3(\mathbb{Z}_2)$
$c = 4$: $\mathrm{End}_{\mathbb{Z}}(C_4 \times C_2)$

D_{12}	
$c = 0$: $\mathbb{Z}_3 \rtimes \mathbb{Z}[C_2]$	$c = 4$: $\mathbb{Z}_4 \times (\mathbb{Z}_2)^{2\times 2}$
$c = 2$: $\mathbb{Z}_2^\varepsilon \times (\mathbb{Z}_2)^{2\times 2}$	$c = 6$: $\mathbb{Z}_2 \times U_2(\mathbb{Z}_3)$
$c = 3$: $U_2(\mathbb{Z}_3)$	

D_{4k} ($k > 3$ & odd)
$c = 0$: $\mathbb{Z}_k \rtimes \mathbb{Z}[C_2]$

In the above table, $\mathbb{Z}_2^\varepsilon = \mathbb{Z}[x]/(x^2)$ as defined before. We denote the ring of 2×2 matrices over $\mathbb{Z}_2$ by $(\mathbb{Z}_2)^{2\times 2}$, the ring of 3×3 upper triangular matrices with entries in $\mathbb{Z}_2$ by $U_3(\mathbb{Z}_2)$, and the ring of 2×2 upper triangular matrices over $\mathbb{Z}_3$ by $U_2(\mathbb{Z}_3)$. Also, recall that $\mathrm{End}_{\mathbb{Z}}(C_4 \times C_2)$ is the ring of endomorphisms of the group $C_4 \times C_2$. We must explain what is meant by the semidirect product of $\mathbb{Z}_k$ and the group algebra $\mathbb{Z}[C_2]$.

Example 5.2. *Recall the semidirect product ring construction of Remark 1.4. Let k be an odd positive integer, $R = \mathbb{Z}_k$, and $S = \mathbb{Z}[C_2] = \{a + bx \mid a, b \in \mathbb{Z}\}$ where $x^2 = 1$ (i.e., the group algebra of C_2 over $\mathbb{Z}$). Next, let $f, g : \mathbb{Z}[C_2] \to \mathbb{Z}_k$ be defined by $f(a + bx) = a + b$ and $g(a + bx) = a - b$ respectively (i.e., f and g are evaluations at $x = 1$ and $x = -1$ respectively). Then it is easy to show that if $r = (1, 1)$ and $s = (0, x)$, then $r^{2k} = 1$, $s^2 = 1$, $(rs)^2 = 1$, and finally $(\mathbb{Z}_k \rtimes \mathbb{Z}[C_2])^\times = \{1, r, \ldots, r^{2k-1}, s, rs, \ldots, r^{2k-1}s\} = D_{4k}$.*

6. Alternating and Symmetric Groups

We have already seen that the center of a group constrains the characteristics available for realization. Since large enough alternating and symmetric groups have trivial centers, it should not be surprising that most cannot be realized at all. In [3], Davis and Occhipinti completely and concisely solve Fuchs' problem for alternating and symmetric groups. Again, instead of reproving their results, we merely relay their conclusions.

Theorem 6.1. *The only realizable (finite) symmetric and alternating groups are S_1, S_2, S_3, S_4, A_1, A_2, A_3, A_4, and A_8.*

Note that $S_1 = A_1 = A_2$ are just the trivial group, $S_2 \cong C_2$, $A_3 \cong C_3$, and $S_3 \cong D_6$ all of which have been discussed before. The remaining groups have trivial centers. Thus they can only be realized in characteristic 2. The following table gives examples of realizations of the symmetric and alternating groups in the only viable characteristics:

S_1	S_2	S_3	S_4
$c=1$: $\{0\}$	$c=0$: $\mathbb{Z}$ $c=3$: $\mathbb{Z}_3$ $c=6$: $\mathbb{Z}_6$	$c=2$: $(\mathbb{Z}_2)^{2\times 2}$	$c=2$: $\mathbb{Z}_2[S_4]/J$
$c=2$: $\mathbb{Z}_2$	$c=2$: $\mathbb{Z}_2^{\varepsilon}$ $c=4$: $\mathbb{Z}_4$		

$A_1 \cong A_2$	A_3	A_4	A_8
$c=1$: $\{0\}$ $c=2$: $\mathbb{Z}_2$	$c=2$: $\mathbb{F}_4$	$c=2$: $\mathcal{O}/2\mathcal{O}$	$c=2$: $(\mathbb{Z}_2)^{4\times 4}$

As before, $\mathbb{Z}_2^{\varepsilon} = \mathbb{Z}[x]/(x^2)$. We let $(\mathbb{Z}_2)^{4\times 4}$ denote the ring of 4×4 matrices over $\mathbb{Z}_2$. To realize S_4, Davis and Occhipinti use a quotient of the group algebra of S_4 over $\mathbb{Z}_2$. Specifically, $\mathbb{Z}_2[S_4]/J$ realizes S_4 if we let $J = (\sigma, \tau)$ where $\sigma = (1)+(24)+(12)(34)+(1234)$ and $\tau = (1)+(12)+(23)+(13)+(123)+(132)$. They then realize A_4 using a quotient of a subring of the quaternion algebra $\mathbb{H}$. In particular, we let $\mathcal{O} = \text{span}_{\mathbb{Z}}\{1, \mathbf{i}, \mathbf{j}, \omega\}$ where $\omega = \dfrac{1}{2}(1+\mathbf{i}+\mathbf{j}+\mathbf{k})$ and quotient by $2\mathcal{O}$. Note that elements of $\mathcal{O}$ are known as Hurwitz quaternions.

7. Dicyclic Groups

In our survery of Fuchs' problem for groups of order at most 15, we have considered all but the dicyclic group of order 8 (also known as the quaternion group of order 8) and the dicyclic group of order 12. In this section we explore the family of dicyclic groups following and extending work found in the last author's Senior Honors Thesis [10] which in turn was inspired by work in the first author's Master's Thesis [1].

For any positive integer n, the dicyclic group of order $4n$ can be presented by $\text{Dic}_{4n} = \langle r, s \mid r^{2n} = 1, s^2 = r^n, rs = sr^{-1}\rangle$. When n is a power of 2, $\text{Dic}_{4n} = Q_{4n}$ (the generalized quaternion group of order $4n$). We note that some authors use names *dicyclic group* and *generalized quaternion group* synonymously even when n is not a power of 2. The smallest dicyclic groups are $\text{Dic}_4 \cong C_4$ (the only Abelian dicyclic group) and $\text{Dic}_8 \cong Q = \{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ (i.e., the group of quaternions of order 8).

One should notice the similarities between the presentations of dicyclic and dihedral groups. While dicyclic and dihedral groups are quite different, they share enough features that many techniques and ideas found in [2] were successfully adapted and used here.

We begin by noting $\text{Dic}_{4n} = \{1, r, \ldots, r^{2n-1}, s, sr, \ldots, sr^{2n-1}\}$ and that one obtains an isomorphism between Dic_{4n} and a subgroup of the unit group of the quaternion algebra $\mathbb{H}$ via the map extending $r \mapsto \exp(2\pi i/2n)$ (a primitive $2n$-th root of unity) and $s \mapsto j$.

The final relation, $rs = sr^{-1}$ implies $rsr = s$ and so $sr = r^{-1}s$ as well. Notice that $(r^k s)^2 = r^k s r^k s = r^k r^{-k} s^2 = r^n \neq 1$, but $(r^k s)^4 = (r^n)^2 = 1$. Therefore, $\langle r\rangle$ is a cyclic subgroup of order $2n$ and each of the remaining elements, $s, rs, \ldots, r^{2n-1}s$, have order 4. Of course, $\text{Dic}_4 = Z(\text{Dic}_4) = \{1, r, s, rs\} = \langle s\rangle \cong C_4$, but when $n > 1$, one has that $Z(\text{Dic}_{4n}) = \{1, r^n\} = \{1, s^2\}$. Either way the center contains the unique element of order 2: $r^n = s^2$. Our observations that for $n > 1$ the center of the group has order two and that it has a unique element of order 2 immediately yield:

Lemma 7.1. *When $n > 1$, it is only possible to realize Dic_{4n} in characteristics $0, 2, 3, 4,$ or 6. Moreover, if we realize a dicyclic group in a characteristic other than 2, then we must have $r^n = s^2 = -1$.*

A pair of simple calculations will rule out characteristics 3 and 6.

Lemma 7.2. *Dicyclic groups cannot be realized in characteristic 3.*

Proof: Suppose R realizes Dic_{4n} and $\text{char}(R) = 3$. Since $\text{char}(R) = 3 \neq 2$, we have that $r^n = s^2 = -1$. Consider $(1+s)^4 = 1 + 4s + 6s^2 + 4s^3 + s^4 = 1 + s + 0 + (-1)s + 1 = 2 = -1$. Therefore, $(1+s)^8 = 1$ and

$(1+s)^4 = -1 \neq 1$. This means that $1+s$ is a unit of order 8. Since $s, rs, \ldots, r^{2n-1}s$ are elements of order 4, we must have $1+s = r^k$ for some k. This implies that $r(1+s) = r^{k+1} = (1+s)r$ and so $rs = sr$. This then implies that $r = r^{-1}$, but then $r^2 = 1$ and so $n = 1$. However, Dic_4 (a cyclic group of order 4) has no elements of order 8. We have reached a contradiction. □

Our argument for characteristic 6 is quite similar.

Lemma 7.3. *Dicyclic groups cannot be realized in characteristic 6.*

Proof: Suppose R realizes Dic_{4n} and $\mathrm{char}(R) = 6$. Again, since $\mathrm{char}(R) = 6 \neq 2$, we have that $r^n = s^2 = -1$. Consider $(2+s)^4 = 2^4 + 4\cdot 2^3 s + 6\cdot 2^2 s^2 + 4\cdot 2s^3 + s^4 = 4 + 2s + 0 + 2(-1)s + 1 = 5 = -1$. Therefore, $(2+s)^8 = 1$ and $(2+s)^4 = -1 \neq 1$. This means that $2+s$ is a unit of order 8. As in the proof above, noting that $2+s$ commutes with r will yield the same contradiction. □

It turns out that characteristics 2 and 4 are viable but only for the first few dicyclic groups. Again we established this using a series of calculations. First, we establish a preliminary identity.

Lemma 7.4. *Suppose $n > 1$ and R realizes Dic_{4n} in characteristic 2. Then for every integer k, we have $r^k + r^{n+k} = 1 + r^n = r^k s + r^{n+k}s$.*

Proof: We observe that in characteristic 2: $(1+r^n)^2 = 1 + 2r^n + r^{2n} = 1 + 1 = 0$. Now notice that $(r^k + 1 + r^n)^2 = r^{2k} + (1+r^n)^2 = r^{2k}$. Therefore, $r^k + 1 + r^n$ is a unit. If $r^k + 1 + r^n = r^\ell s$ for some integer ℓ, we would have that $r^\ell s$ is a polynomial in r and thus commutes with r. This would imply that $r^{\ell+1}s = r^{\ell-1}s$ and consequently $r = r^{-1}$ so that $r^2 = 1$ and so $n = 1$ (contradiction). Thus we must have that $r^k + 1 + r^n = r^\ell$ for some integer ℓ so that $1 + r^n = r^k + r^\ell$. Squaring both sides yields: $0 = r^{2k} + r^{2\ell}$. Since $r^{2k} = r^{2\ell}$ and r is of order $2n$, we must have $2k \equiv 2\ell \bmod 2n$ and so $k \equiv \ell \bmod n$. Notice that we cannot have $k = \ell$ since otherwise $1 + r^n = r^k + r^k = 0$ and so $r^n = 1$ (contradiction). Therefore, $\ell = k + n$. We have established that $r^k + r^{k+n} = 1 + r^n$.

Likewise consider $r^k s + 1 + r^n$. Then $(r^k s + 1 + r^n)^2 = (r^k s)^2 + (1+r^n)^2 = r^n + 0 = r^n$. Thus $r^k s + 1 + r^n$ is a unit of order 4. Next, if $r^k s + 1 + r^n = r^\ell$, we would have $r^k s = 1 + r^n + r^\ell$. This again forces s to commute with r and thus implies $n = 1$ (contradiction). Therefore, $r^k s + 1 + r^n = r^\ell s$ for some integer ℓ. Thus $1 + r^n = r^k s + r^\ell s$. Squaring both sides yields $0 = r^n + r^{k-\ell}s^2 + r^{\ell-k}s^2 + r^n$. Thus $0 = r^{k-\ell+n} + r^{\ell-k+n}$ which implies that $k - \ell \equiv \ell - k \bmod 2n$ (i.e., $2k \equiv 2\ell \bmod 2n$). Therefore, $k \equiv \ell \bmod n$. However, $k \neq \ell$ since otherwise we would have $1 + r^n = r^k s + r^k s = 0$ so that $r^n = 1$ (contradiction). Therefore, $\ell = k + n$. We have established that for any integer k, $1 + r^n = r^k s + r^{k+n}s$. □

Lemma 7.5. *If R realizes Dic_{4n} in characteristic 2, then $n = 1$ or 2.*

Proof: Suppose $n > 1$. Consider the element $x = (r + r^{n-1})(1+s) = r + r^{n-1} + rs + r^{n-1}s$. Keeping in mind that $sr^k = r^{-k}s$, $s^2 = r^n$, and $r^k = r^\ell$ if $k \equiv \ell \bmod 2n$, we have $x^2 = r^2 + r^n + r^2 s + r^n s + r^n + r^{2n-2} + r^n s + r^{2n-2}s + s + r^{n+2}s + r^n + r^2 + r^{n-2}s + s + r^{2n-2} + r^n = 2r^2 + 4r^n + r^2 s + 2r^n s + 2r^{2n-2} + r^{2n-2}s + 2s + r^{n+2}s + r^{n-2}s = r^2 s + r^{n+2}s + r^{n-2}s + r^{2n-2}s$. Using the lemma above, we have $x^2 = 1 + r^n + 1 + r^n = 0$. Therefore, $u = 1 + x$ is a unit of order 1 or 2 since $u^2 = 1^2 + x^2 = 1 + 0 = 1$. In particular, $u = 1$ or $u = r^n$. Either way, $u = 1 + r + r^{n-1} + rs + r^{n-1}s$ is a power of r and so $rs + r^{n-1}s$ commutes with r. Therefore, $r^2 s + r^n s = r(rs + r^{n-1}s) = (rs + r^{n-1}s)r = s + r^{n-2}s$ and so $r^2 + r^n = 1 + r^{n-2}$. We now have $1 + r^n = r^2 + r^{n-2}$, but the lemma above gives us $1 + r^n = r^2 + r^{n+2}$. Therefore, $r^{n-2} = r^{n+2}$ and so $r^4 = 1$. This implies $n \leq 2$. □

We have the same result for characteristic 4. However, our calculations are a little easier this time. We begin by establishing some helpful identities.

Lemma 7.6. *Suppose R realizes Dic_{4n} in characteristic 4. If $x \in R$, then either $2x = 0$ or $2x = 2$. Moreover, if $x \in R^\times$, then $2x = 2$.*

Proof: Since $\mathrm{char}(R) \neq 2$, we have $r^n = s^2 = -1$. Let $x \in R$. We have $(1+2x)^2 = 1 + 4x + 4x^2 = 1$. Thus $1 + 2x$ is always a unit of order 1 or 2. In particular, $1 + 2x = 1$ or $1 + 2x = -1$. Therefore, either $2x = 0$ or $2x = 2$. Moreover, if $x \in R^\times$, then $2x = 0$ would imply that $2 = 0x^{-1} = 0$ so that $\mathrm{char}(R) \neq 4$

(contradiction). Thus for all $x \in R^\times$, we have $2x = 2$. □

Lemma 7.7. *If R realizes Dic_{4n} in characteristic 4, then $n = 1$ or 2.*

Proof: Again, we note that since $\mathrm{char}(R) \neq 2$, we have $r^n = s^2 = -1$. Consider $x = r + r^{n-1} + rs + r^{n-1}s$. Keeping in mind that $r^n = -1$, we have $x^2 = r^2 + r^n + r^2 s + r^n s + r^n + r^{2n-2} + r^n s + r^{2n-2}s + s + r^{n+2}s + r^n + r^2 + r^{n-2}s + s + r^{2n-2} + r^n = r^2 - 1 + r^2 s - s - 1 + r^{2n-2} - s - r^{n-2}s + s - r^2 s - 1 + r^2 + r^{n-2}s + s + r^{2n-2} - 1 = 2r^2 + 2r^{2n-2} = 2 + 2 = 0$ noting that r^2 and r^{2n-2} are units and using our lemma above.

Next, notice that $2x = 2r + 2r^{n-1} + 2rs + 2r^{n-1}s = 2 + 2 + 2 + 2 = 0$ where repeatedly use our lemma above after noting that r, r^{n-1}, rs, and $r^{n-1}s$ are units. Therefore, $(1 + x)^2 = 1 + 2x + x^2 = 1 + 0 + 0 = 1$ so that $1 + x$ is a unit of order 1 or 2. In particular, either $1 + x = 1$ or $1 + x = -1$ so that $x = 0$ or $x = 2$. Therefore, $rs + r^{n-1}s$ equals $-r - r^{n-1}$ or $-r - r^{n-1} + 2$. Either way, $rs + r^{n-1}s = (r + r^{n-1})s$ is a polynomial in r and thus must commute with r. However, $(r^2 - 1)s = r(rs + r^{n-1}s) = (rs + r^{n-1}s)r = (1 + r^{n-2})s$. Thus $r^2 - 1 = 1 + r^{n-2}$ and so $r^2 - 1 = 1 - r^{-2}$. Therefore, $r^2 = 2 - r^{-2}$ and so $r^4 = (2 - r^{-2})^2 = 4 - 4r^{-2} + r^{-4}$. Thus $r^8 = 1$ and so the order of r divides 8. Suppose the order of r is 8. Then $r^4 = -1$. We already know $r^2 = 2 - r^{-2}$. Multiplying by r^2 yields $-1 = r^4 = 2r^2 - 1$ so $2r^2 = 0$. But r^2 is a unit, so this contradicts our lemma above. Therefore, the order of r must be 2 or 4 (i.e., $n = 1$ or 2). □

We have now shown that for $n > 2$, Dic_{4n} cannot be realized in a non-zero characteristic. The results for non-zero characteristic above are sharp.

Example 7.8. *$\mathrm{Dic}_4 \cong C_4$ can be realized by $\mathbb{Z}[i]$, $\mathbb{Z}_2^\delta = \mathbb{Z}_2[x]/(x^3)$, $\mathrm{Gil} = \mathbb{Z}_4[x]/(2x, x^2 + 2)$, $\mathbb{Z}_5$, and $\mathbb{Z}_{10}$ in characteristics 0, 2, 4, 5, and 10 respectively. All other characteristics are impossible.*

Example 7.9. *Dic_8 (the quaternion group of order 8) can be realized by $\mathbb{Z}[i, j, k]$, $RD_2 = \mathbb{Z}_2[\mathrm{Dic}_8]/(1 + s + r + rs)$, and $RD_4 = \mathbb{Z}_4[\mathrm{Dic}_8]/(1 + s + r + rs, 1 + s^2)$ in characteristics 0, 2, and 4 respectively. All other characteristics are impossible.*

Note that it is easy to check that $(\mathbb{Z}[i, j, k])^\times \cong \mathrm{Dic}_8$. The other realizations (i.e., the quotients of groups algebras RD_2 and RD_4) were checked using [7] in [10].

Corollary 7.10. *Suppose a finite ring R realizes Dic_{4n}. Then either $n = 1$ and $\mathrm{char}(R) \in \{2, 4, 5, 10\}$ or $n = 2$ and $\mathrm{char}(R) \in \{2, 4\}$. Conversely, there are rings realizing Dic_4 in characteristics 2, 4, 5, and 10 as well as rings realizing Dic_8 in characteristics 2 and 4.*

This solves Fuchs' problem restricted to finite rings, moving to characteristic 0, we no longer have a complete solution. However, we do have some partial results. First, recall that if we let $\omega_{2n} = \exp(2\pi i/2n)$ where n is some positive integer, then the cyclotomic integer ring $\mathbb{Z}[\omega_{2n}]$ has infinitely many units if and only if $n > 3$ (see Appendix A). Therefore, $\mathbb{Z}[\omega_{2n}, j]$ (a subring of the quarternion algebra) also has infinitely many units. Thus we cannot directly use our construction of dicyclic groups inside $\mathbb{H}$ to realize them in characteristic 0 (at least when $n > 3$).

On the other hand, when $n = 1$, 2, and 3, this construction yields exactly what we want.

Example 7.11. *Consider $\omega_2 = -1$ and $\mathbb{Z}[\omega_2, j] = \mathbb{Z}[j]$. This is the Gaussian intergers which realize $\mathrm{Dic}_4 \cong C_4$ in characteristic 0. Likewise, consider $\omega_4 = i$ and $\mathbb{Z}[\omega_4, j] = \mathbb{Z}[i, j] = \{n_1 + n_2 i + n_3 j + n_4 k \mid n_1, n_2, n_3, n_4 \in \mathbb{Z}\}$ (i.e., the Lipschitz integers). This ring realizes Dic_8 (the quaternion group of order 8) in characteristic 0.*

The final workable case is that of $\omega_6 = \dfrac{1 + \sqrt{3}i}{2}$ and $\mathbb{Z}[\omega_6, j]$. Recall that $\mathbb{Z}[\omega_6]$ are called Eisenstein integers. We might call our ring $\mathbb{Z}[\omega_6, j]$ the Eisenstein quaternions.

Example 7.12. *The Eisenstein quaternions realize Dic_{12} in characteristic 0.*

For simplicity let $\omega = \omega_6 = \dfrac{1 + \sqrt{3}i}{2}$. Notice that $\omega^2 = \dfrac{-1 + \sqrt{3}i}{2} = -1 + \omega$ and $\omega^5 = \overline{\omega} = \dfrac{1 - \sqrt{3}i}{2}$. Thus $\mathbb{Z}[\omega, j] = \{n_1 + n_2\omega + n_3 j + n_4\omega j \mid n_1, n_2, n_3, n_4 \in \mathbb{Z}\}$ and this ring is closed under quaternion conjugation.

Next, $q = n_1 + n_2\omega + n_3 j + n_4\omega j = \left(n_1 + \dfrac{n_2}{2}\right) + \dfrac{n_2\sqrt{3}}{2}i + \left(n_3 + \dfrac{n_4}{2}\right)j + \dfrac{n_4\sqrt{3}}{2}k$ so that the norm of q is $N(q) = \left(n_1 + \dfrac{n_2}{2}\right)^2 + \dfrac{3n_2^2}{4} + \left(n_3 + \dfrac{n_4}{2}\right)^2 + \dfrac{3n_4^2}{4} = n_1^2 + n_1 n_2 + n_2^2 + n_3^2 + n_3 n_4 + n_4^2$. In particular, $N(q) \in \mathbb{Z}_{\geq 0}$.

If q is a unit, then $N(q)N(q^{-1}) = N(qq^{-1}) = N(1) = 1$, but since $N(q)$ is a non-negative integer, we must have $N(q) = 1$. Conversely, if $N(q) = 1$, then $q\overline{q} = \overline{q}q = N(q) = 1$ so that $q^{-1} = \overline{q}$ and thus q is a unit.

Notice that $n_1^2 + n_1n_2 + n_2^2 = \left(n_1 + \frac{n_2}{2}\right)^2 + \frac{3n_2^2}{4} \geq 0$ and likewise $n_3^2 + n_3n_4 + n_4^2 \geq 0$. Therefore, $n_1^2 + n_1n_2 + n_2^2 = 1$ and $n_3^2 + n_3n_4 + n_4^2 = 0$ or $n_1^2 + n_1n_2 + n_2^2 = 0$ and $n_3^2 + n_3n_4 + n_4^2 = 1$. If $n_1^2 + n_1n_2 + n_2^2 = \left(n_1 + \frac{n_2}{2}\right)^2 + \frac{3n_2^2}{4} = 0$, we must have $n_1 = n_2 = 0$ (and similarly for n_3 and n_4). Notice that $1 = \left(n_1 + \frac{n_2}{2}\right)^2 + \frac{3n_2^2}{4} \geq \frac{3n_2^2}{4}$ implies $|n_2| < 2$ so that $n_2 = 0, \pm 1$. By symmetry the same is true for n_1. We get that the only solutions of $n_1^2+n_1n_2+n_2^2 = 1$ are $(n_1, n_2) = (\pm 1, 0)$, $(0, \pm 1)$, and $(\pm 1, \mp 1)$. The same applies to n_3 and n_4. Putting this altogether, we have that $\mathbb{Z}[\omega, j]^\times = \{\pm 1, \pm\omega, \pm(1-\omega), \pm j, \pm\omega j, \pm(1-\omega)j\}$. This unit group is precisely Dic_{12}.

If we take into account the realization results of all groups of orders 12 or less, we arrive at the following interesting result:

Corollary 7.13. *The dicyclic group of order 12, Dic_{12}, is the smallest realizable group that cannot be realized by a finite ring (i.e., Dic_{12} is the smallest group realizable only in characteristic 0).*

Adapting a construction found in [2], we can construct infinitely many other dicyclic groups in characteristic 0.

Example 7.14. *Again recall the semidirect product ring construction found in Remark 1.4. Let k be a positive integer such that every prime factor of k is congruent to 1 modulo 4. This implies that $\mathbb{Z}_k^\times$ contains an element τ such that $\tau^2 = -1$ and that k is odd.*

Consider $f, g : \mathbb{Z}[i] \to \mathbb{Z}_k$ defined by $f(x + yi) = x + y\tau$ and $g(x + yi) = x - y\tau$ (i.e., evaluation at $\pm\tau$). Equipped with these ring homomophisms, we get that $\mathbb{Z}_k \rtimes \mathbb{Z}[i]$ is a ring with multiplication: $(m, a + bi)(n, c + di) = (m(c + d\tau) + n(a - b\tau), (a + bi)(c + di))$. Our units consist of the set $(\mathbb{Z}_k \rtimes \mathbb{Z}[i])^\times = \{(m, \pm 1), (m, \pm i) \mid m \in \mathbb{Z}_k\}$.

A simple inductive argument shows that $(-1, -1)^\ell = ((-1)^\ell \ell, (-1)^\ell)$. In particular, $(-1, -1)^k = ((-1)^k k, (-1)^k) = (0, -1)$ since k is odd and also $(-1, -1)^{2k} = (0, 1)$ so that $(-1, -1)^{-1} = (-1, -1)^{2k-1} = ((-1)^{2k-1}(2k - 1), (-1)^{2k-1}) = (1, -1)$. It also follows that $(-1, -1)^k = (0, -1) = (0, i)^2$. Next, notice that $(-1, -1)(0, i) = ((-1)\tau + 0(-1), (-1)i) = (-\tau, -i)$ and $(0, i)(-1, -1)^{-1} = (0, i)(1, -1) = (0(-1), 1(-\tau), i(-1)) = (-\tau, -i)$.

In summary, if we let $r = (-1, -1)$, let $s = (0, i)$, and recall that $1 = (0, 1)$, then we have shown $r^{2k} = 1$, $r^k = s^2$, and $rs = sr^{-1}$. Since there are exactly $4k$ units in this ring, we have established $(\mathbb{Z}_k \rtimes \mathbb{Z}[i])^\times \cong \mathrm{Dic}_{4k}$.

In particular, we can realize dicyclic groups such as Dic_{20}, Dic_{52}, and Dic_{100}, but this construction and our results say nothing about Dic_{28} and Dic_{36}. On the other hand, we can rule out some dicyclic groups altogether using Proposition 1.3 . In the context of dicyclic groups this proposition tells us the following:

Proposition 7.15. *If we can realize Dic_{4n} in characteristic c, then we can realize C_{2n} in characteristic c.*

Example 7.16. *The dicyclic group Dic_{16} is not realizable. For sake of contradiction, suppose Dic_{16} is realized by R. Then by Corollary 7.10, we must have $\mathrm{char}(R) = 0$. By the above proposition, we can then realize C_8 in characteristic 0, but according to Lemma 3.3 this is not possible.*

Let us extend this example by relying on the main result from [9]. In particular, Theorem 3.1 says that while any cyclic group of twice an odd integer order can be realized, if the order is divisible by 4, realizability is not guaranteed. In fact, Theorem 5.1 and Corollary 5.17 in [5], place several restrictions on n. In particular, we find the cyclic groups of orders divisible by 8 are not realizable in characteristic 0. Since Dic_{4n} must be realized in characteristic 0 for $n > 2$ and realizing Dic_{4n} in characteristic 0 implies we can realize C_{2n} in characteristic 0, we have the following:

Proposition 7.17. *If n is divisible by 4, then Dic_{4n} cannot be realized.*

We provide the following table of realizations of dicyclic groups Dic_{4n} in all viable characteristics for $n \leq 5$ as well as partial results for some larger n:

$\mathrm{Dic}_4 \cong C_4$	$\mathrm{Dic}_8 \cong Q$	Dic_{12}	Dic_{16}	Dic_{20}
$c=0$: $\mathbb{Z}[i]$ $\quad c=5$: $\mathbb{Z}_5$	$c=0$: $\mathbb{Z}[i,j,k]$	$c=0$: $\mathbb{Z}[\omega_6,j]$	None	$c=0:$ $\mathbb{Z}_5 \rtimes \mathbb{Z}[i]$
$c=2$: $\mathbb{Z}_2^\delta$ $\quad c=10$: $\mathbb{Z}_{10}$	$c=2$: RD_2			
$c=4$: Gil	$c=4$: RD_4			

Dic_{4n} where every prime dividing n is congruent to 1 mod 4	Dic_{4n} where n is divisible by 4
$c=0$: $\mathbb{Z}_n \rtimes \mathbb{Z}[i]$	None

Appendix A. Roots of Unity

In this appendix, we collect some facts about roots of unity. In particular, we look at the units of the rings generated a root of unity (i.e., rings of cyclotomic integers).

Let m be a positive integer and $\omega_m = \exp(2\pi i/m)$, so ω_m is a primitive m-th root of unity. Note that if m is odd, $(-\omega_m)^m = (-1)^m(\omega_m)^m = -1$, so $-\omega_m$ is a $(2m)$-th primitive root of unity. In particular, when m is odd, $\mathbb{Z}[\omega_m] = \mathbb{Z}[\omega_{2m}]$.

The first few such rings are familiar. When $m = 1$ or 2 we have the integers: $\mathbb{Z}[\omega_1] = \mathbb{Z}[\omega_2] = \mathbb{Z}[-1] = \mathbb{Z}$ where $\mathbb{Z}^\times = \{\pm 1\} \cong C_2$. Selecting $m = 4$ yields the Gaussian integers: $\mathbb{Z}[\omega_4] = \mathbb{Z}[i]$ where $\mathbb{Z}[i]^\times = \{\pm 1, \pm i\} \cong C_4$. Finally, selecting $m = 3$ or 6 yields the possibly less familiar Eisenstein integers $\mathbb{Z}[\omega_3] = \mathbb{Z}[\omega_6]$. Note that $\omega_3 = \dfrac{-1+i\sqrt{3}}{2}$ and $(\omega_3)^2 + \omega_3 + 1 = 0$ so $(\omega_3)^2 = -\omega_3 - 1$. In particular, we can reduce any quadratic (or higher) power of ω_3. Thus $\mathbb{Z}[\omega_3] = \{a + b\omega_3 \mid a, b \in \mathbb{Z}\}$.

To determine the group of units of $\mathbb{Z}[\omega_3]$, we use the norm: $N(z) = z\overline{z} = |z|^2$ (i.e., the complex modulus squared). For any $z = a+b\omega_3 \in \mathbb{Z}[\omega_3]$ where $a, b \in \mathbb{Z}$, we have $N(z) = N(a+b\omega_3) = N\left(\left(-a + \frac{b}{2}\right) + \frac{b\sqrt{3}}{2}i\right) = \left(-a + \frac{b}{2}\right)^2 + \frac{3b^2}{4} = a^2 - ab + b^2$. In particular, $N(a + b\omega_3)$ is a non-negative integer for any $a + b\omega_3 \in \mathbb{Z}[\omega_3]$. Since the norm's values are non-negative integers and the norm is multiplicative, a standard argument implies $a + b\omega_3$ is a unit if and only if $N(a + b\omega_3) = 1$.

Suppose $z = a + b\omega_3$ is a unit of $\mathbb{Z}[\omega_3]$ (where $a, b \in \mathbb{Z}$). Thus $\left(-a + \frac{b}{2}\right)^2 + \frac{3b^2}{4} = a^2 - ab + b^2 = 1$ so that $\frac{3b^2}{4} \leq 1$. Thus, we must have $|b| < 2$. We also have $a^2 - ab + b^2 = \frac{3a^2}{4} + \left(-b + \frac{a}{2}\right)^2$, so $|a| < 2$ as well. Thus $a, b \in \{0, \pm 1\}$. The only solutions are $a = 0$ with $b = \pm 1$, $a = \pm 1$ with $b = 0$, or $a = b = \pm 1$. These are precisely the sixth roots of unity: $\{\pm 1, \pm\omega_3, \pm(1 + \omega_3)\}$. Thus $\mathbb{Z}[\omega_6]^\times = \mathbb{Z}[\omega_3]^\times = \{1, \omega_6, \dots, (\omega_6)^5\} \cong C_6$.

Thus $\mathbb{Z}[\omega_m]$ has a finite group of units when $m = 1, 2, 3, 4$ or 6. It turns out that all other rings of cyclotomic integers have infinitely many units. We confirm this with a concrete calculation.

Let $m > 1$ so that $\omega_m \neq 1$. Let k be an integer such that $1 < k < m$ and $\gcd(k, m) = 1$. Then there exists $x, y \in \mathbb{Z}$ such that $kx = my + 1$. Notice that $k(x + m\ell) = m(y + k\ell) + 1$. Thus we may assume that $x, y > 0$. Also, ω_m is primitive implies $(\omega_m)^k \neq 1$ (since $k < m$). Consider the following element:

$$u = \frac{(\omega_m)^k - 1}{\omega_m - 1} = (\omega_m)^{k-1} + \cdots + \omega_m + 1 \in \mathbb{Z}[\omega_m]$$

Then we have that:

$$u^{-1} = \frac{\omega_m - 1}{(\omega_m)^k - 1} = \frac{(\omega_m)^{my+1} - 1}{(\omega_m)^k - 1} = \frac{(\omega_m)^{kx} - 1}{(\omega_m)^k - 1} = (\omega_m)^{(x-1)k} + \cdots + (\omega_m)^{x-1} + 1 \in \mathbb{Z}[\omega_m]$$

Thus we have demonstrated that $u \in \mathbb{Z}[\omega_m]^\times$.

Since when m is odd, $\mathbb{Z}[\omega_m] = \mathbb{Z}[\omega_{2m}]$, we only need to consider even m. Consider $m = 2\ell$ where ℓ is odd and $\ell > 3$. Then $\ell - 2$ is odd and $m = 2\ell$ and $k = \ell - 2$ are relatively prime. Thus $u = (\omega_m)^{\ell-3} + \cdots + \omega_m + 1$ is a unit in $\mathbb{Z}[\omega_m]$. Likewise, consider $m = 2\ell$ where ℓ is even and $\ell > 2$. Then $\ell - 1$ is odd and $m = 2\ell$ and $k = \ell - 1$ are relatively prime. Thus $u = (\omega_m)^{\ell-2} + \cdots + \omega_m + 1$ is a unit in $\mathbb{Z}[\omega_m]$.

In either case, we have a unit u in $\mathbb{Z}[\omega_m]$ as a sum of 1 plus other roots of unity in the upper-half plane of $\mathbb{C}$ and thus u lies outside the unit circle in $\mathbb{C}$. Therefore, u itself is not a root of unity. In fact, $N(u) > 1$ so that u is an element of infinite multiplicative order in $\mathbb{Z}[\omega_m]^\times$. Therefore, $\mathbb{Z}[\omega_m]^\times$ is infinite for $m = 5$ or $m > 6$.

Consequently, while we have a copy of Dic_{4n} within the subring $\mathbb{Z}[\omega_{2n}, j]$ of the quaternions, this ring contains infinitely many units when $n > 3$ and thus cannot directly realize Dic_{4n} for $n > 3$.

Appendix B. Ring Constructions

In this appendix we calculate unit groups of various rings used in realizations throughout the paper.

Lemma B.1. *Let p be a prime. The unit group of $\mathbb{Z}_p^\varepsilon = \mathbb{Z}_p[x]/(x^2)$ is $C_{p(p-1)}$.*

Proof: We work mod (x^2). Every element is equivalent to some $a + bx$ where $a, b \in \mathbb{Z}_p$ are uniquely determined.

First, suppose that $a \neq 0$. Then a^{-1} exists in $\mathbb{Z}_p$. We have $(a+bx)(a^{-1}-a^{-2}bx) = 1+(-a^{-1}b+a^{-1}b)x+a^{-2}b^2x^2 = 1$. On the other hand, if $a = 0$, $bxbx = b^2x^2 = 0$ so either bx is zero or a zero divisor (i.e., not a unit). We know see that the units of $\mathbb{Z}_p^\varepsilon$ are precisely the elements $a+bx$ with $a \neq 0$. Thus there are exactly $p(p-1)$ units.

Since p is prime, there exists some $\zeta \in (\mathbb{Z}_p)^\times$ such that $(\mathbb{Z}_p)^\times = \langle\zeta\rangle$. A simple inductive argument shows that $(\zeta+x)^k = \zeta^k + k\zeta^{k-1}x$ for any positive integer k. Notice that $k\zeta^{k-1} = 0$ requires $k = 0 \pmod p$. Since p and $p-1$ are relatively prime, the first power k such that $\zeta^k = 1$ and $k\zeta^{k-1} = 0$ is $k = p(p-1)$. Therefore, $\zeta + x$ has order $p(p-1)$ and thus generates the group of units. □

Lemma B.2. *The unit group of $\mathbb{Z}_2^\delta = \mathbb{Z}_2[x]/(x^3)$ is C_4.*

Proof: We work mod (x^3). Every element is equivalent to some $a+bx+cx^2$ where $a, b, c \in \mathbb{Z}_2$ are uniquely determined.

Notice that if $a = 0$, then $(bx + cx^2)x^2 = 0$ so that $bx + cx^2$ is either zero or a zero divisor (i.e., not a unit). On the other hand, we have $1 + x$, $(1 + x)^2 = 1 + x^2$, $(1 + x)^3 = 1 + x + x^2$, $(1 + x)^4 = 1$. Thus $(\mathbb{Z}_2^\delta)^\times = \langle 1 + x\rangle \cong C_4$. □

Lemma B.3. *The unit group of* $\mathrm{Gil} = \mathbb{Z}_4[x]/(2x, x^2 + 2)$ *is* C_4.

Proof: Using the relation $x^2 + 2$ (i.e., $x^2 = 2$), we only need to consider representatives of the form $a + bx$ where $a, b \in \mathbb{Z}_4$. In addition, we have $2x = 0$, so we can require $b = 0$ or 1. Thus $\mathrm{Gil} = \{0, 1, 2, 3, x, 1 + x, 2 + x, 3 + x\}$ is a ring with 8 elements.

Notice that $2(x) = 2x = 0$ and $2(2 + x) = 0 + 2x = 0$ thus 2, $2x$, and $2 + x$ are zero divisors. Next, $(1+x)^2 = 1+2x+x^2 = 1+0+2 = 3$, $(1+x)^3 = (1+x)3 = 3+3x = 3+x$, and $(1+x)^4 = (1+x)(3+x) = 3 + 4x + x^2 = 3 + 0 + 2 = 1$. Therefore, $\mathrm{Gil}^\times = \{1, 1 + x, 3, 3 + x\} = \langle 1 + x\rangle \cong C_4$. □

Lemma B.4. *Let m be an odd positive integer. The unit group of $PS_m = \mathbb{Z}[x]/(mx, x^2)$ is C_{2m}.*

Proof: We work mod (x^2) and notice that the relation $2m$ implies every element can be uniquely represented as $a + bx$ where $a \in \mathbb{Z}$ and $b \in \{0, 1, \dots, m - 1\}$.

Let $s, t \in PS_m$ so that $s = a + bx$ and $t = c + dx$ for some $a, c \in \mathbb{Z}$ and $b, d \in \{0, 1, \dots, m - 1\}$. We calculate: $(a + bx)(c + dx) = ac + (ad + bc)x + dbx^2 = ac + (ad + bc \bmod m)x$. If we wish $st = 1$, then we must have $ac = 1$. Therefore, units must be of the form $\pm1 + bx$.

Notice that $(1+bx)(1+(m-b)x) = 1+(b+m-b)x+b(m-b)x^2 = 1+mx = 1$ and $(-1+bx)(-1+(m-b)x) = 1 + (-b - m + b)x + b(m - b)x^2 = 1 - mx = 1$. So $(PS_m)^\times = \{\pm1 + bx \mid b \in \mathbb{Z} \text{ such that } 0 \leq b < m\}$. Also, $(-1 + x)^k = (-1)^k + (-1)^{k-1}kx = 1$ only if k is an even multiple of m. Thus $-1 + x$ has order $2m$. Therefore, $(PS_m)^\times = \langle -1 + x\rangle \cong C_{2m}$. □

Lemma B.5. *The unit group of* $\mathrm{M} = \mathbb{Z}_8[x]/(x^2 + 1, 2x - 2)$ *is* $C_2 \times C_4$.

Proof: The relation $x^2 + 1$ allows us to reduce elements to those of the form $a + bx$ and the second relation $2x - 2$ allows us to assume $b = 0$ or 1. Therefore, $\mathrm{M} = \{a + bx \mid a \in \mathbb{Z}_8 \text{ and } b = 0 \text{ or } 1\}$.

Notice that $(2k + 1 + x)4 = 4(2k + 1) + 4x = 4(2k + 1) + 4 = 8(k + 1) = 0$ for any k since $2x = 2$. Therefore, it follows that 0, 2, 4, 6, $1 + x$, $3 + x$, $5 + x$, and $7 + x$ are not units. On the other hand, for any k, $(2k + x)^2 = 4k^2 + 4kx + x^2 = 4k^2 + 4k - 1 = 4k(k + 1) - 1 = -1$ since $2x = 2$, $x^2 = -1$, and $4k(k + 1)$ is a multiple of 8. Thus $(2k + x)^4 = (-1)^2 = 1$, so x, $2 + x$, $4 + x$, and $6 + x$ are units of order 4. Since 1, 3, 5, and 7 are units, we have that $\mathrm{M}^\times = \{1, 3, 5, 7, x, 2 + x, 4 + x, 6 + x\}$. Moreover, this is an Abelian group of order 8. It has too many elements of order 4 to be cyclic. Thus $\mathrm{M}^\times \cong C_2 \times C_4$. □

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