# Solving $1^{st}$-Order ODEs using Symmetry Methods

Hannah Gilmore $\Longleftrightarrow$ gilmorehf@appstate.edu
Michael Kelley $\Longleftrightarrow$ kelleyma1@appstate.edu
Hadi Morrow $\Longleftrightarrow$ alsoqih@appstate.edu
Huy Tu $\Longleftrightarrow$ tuhq@appstate.edu

Thursday, April 26, 2012

# How Do You Solve “First-Order ODEs” Using Symmetry?

An ordinary differential equation (ODE) is an equation involving (ordinary) derivatives.

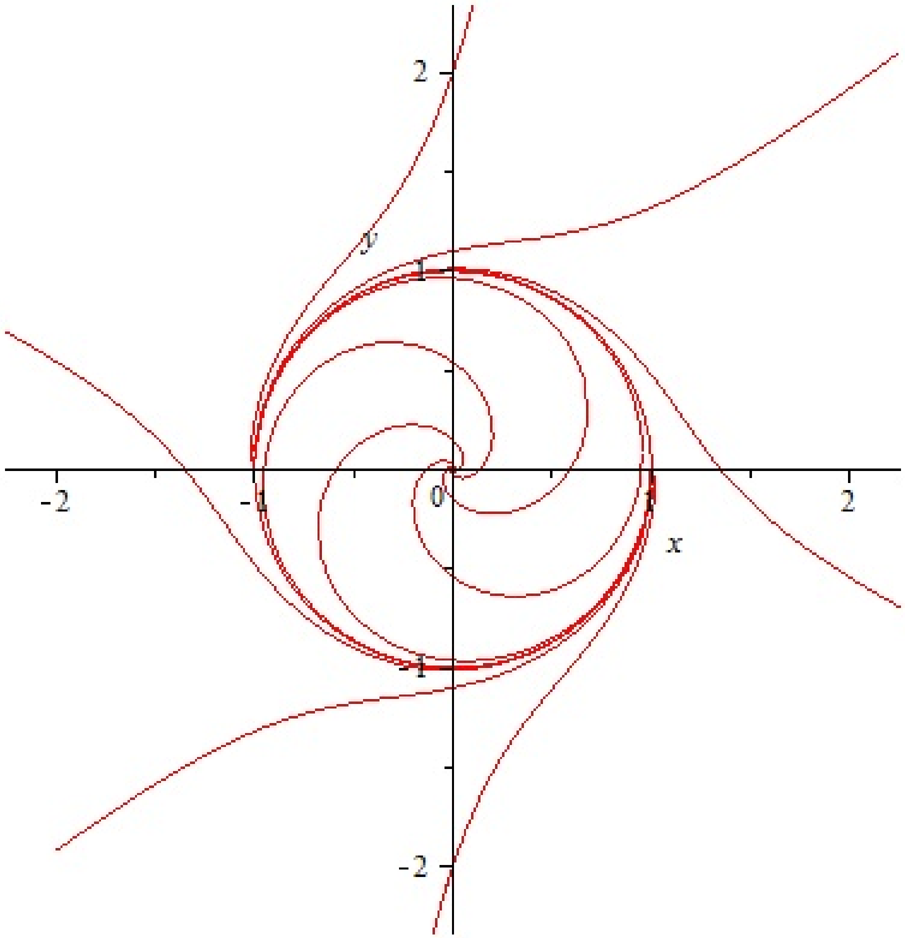
A solution of an ODE is a function such that the ODE is satisfied when the solution and its derivatives are plugged into the equation.

The order of an ODE is the highest order derivative appearing in the equation.

**Example:** $y'' + y = 0$ is a second order ODE. $y = \sin(x)$ is a solution and so is $y = \cos(x)$. In fact, for any choice of real numbers $C_1$ and $C_2$, we have that $y = C_1 \sin(x) + C_2 \cos(x)$ is a solution.

We looked at symmetry methods for solving first order ODEs. These same techniques can be extended to solve higher order ODEs and PDEs (partial differential equations).

# An Equation with Rotational Symmetry



**Example:**

$$y' = \frac{y^3+x^2y-y-x}{x^3+xy^2+y-x}$$

The figure on the left shows several solutions of this equation. We can see that they have an obvious rotational symmetry. It turns out that this can be detected by looking at the form of the equation.

Once this is known, we can easily cook up an integrating factor and solve the equation.

# The Set Up

We start with the first order ODE: $\frac{dy}{dx} = \frac{B(x,y)}{A(x,y)}$.

First, we convert this equation to its Pfaffian form:

$$-Bdx + Ady = 0$$

If $-B_y = -\dfrac{\partial B}{\partial y} = \dfrac{\partial A}{\partial x} = A_x$, then this would be an exact differential equation which could easily be solved. Our goal is to find an integrating factor to make this equation exact so we can solve it.

# Finding an Integrating Factor

Let $\Psi(x, y) = \psi$ be an implicitly defined solution (so $\psi$ is a constant depending on some initial condition).

So $\Psi_x\, dx + \Psi_y\, dy = \cancelto{0}{d\psi}$ or equivalently $\Psi_x + \Psi_y \frac{dy}{dx} = 0$.

Suppose $y(x)$ is an explicit solution: $\Psi(x, y(x)) = \Psi_x \cancelto{1}{\frac{dx}{dx}} + \Psi_y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{B(x,y)}{A(x,y)}$ from original equation thus,

$\Psi_x(x, y) + \Psi_y(x, y)\frac{B(x,y)}{A(x,y)} = 0$ and so

$$\boxed{A(x, y)\Psi_x(x, y) + B(x, y)\Psi_y(x, y) = 0}$$

# Symmetries of Solutions

In general, if $\mathcal{F}$ is a (structure preserving) map or operation such that $\mathcal{F}$ sends some object X to itself, then we say $\mathcal{F}$ is a symmetry of X.

A symmetry of a differential equation is a mapping from a collection of functions to itself which sends solutions (of that ODE) to solutions.

Let $\tilde{x} = F(s, x, y)$ and $\tilde{y} = G(s, x, y)$ such that for each real number $s$, $\tilde{y}(\tilde{x})$ is a solution whenever $y(x)$ is a solution. In other words, for each choice of $s$, $F$ and $G$ define a symmetry of our ODE. [Secretly we are defining a 1-parameter Lie group.]

Moreover, we assume that $F(0, x, y) = x$ and $G(0, x, y) = y$, so $s = 0$ corresponds to the identity map (this sends each solution to itself).

# Tangents of Symmetries

Since $\Psi(x,y)=\psi$ is a solution, then $\Psi(\tilde{x},\tilde{y})=\psi(s)$ is a solution (for possibly some other choice of constant $\psi(s)$). Since $s=0$ corresponds to the identity, $\Psi(x,y)=\psi=\psi(0)$. In general, $\Psi\Big(F(s,x,y),G(s,x,y)\Big)=\psi(s)$.

Expand this equation in terms of its Maclaurin series (in $s$):

$$\begin{array}{rcr} \Psi\big(F(s,x,y),G(s,x,y)\big)\Big|_{s=0} & & \psi(0) \\ +\Big[\Psi_x(\ldots)F_s(s,x,y)+\Psi_y(\ldots)G_s(s,x,y)\Big]\Big|_{s=0}\; s & = & \psi'(0) \\ +O(s^2) & & +O(s^2) \end{array}$$

$$\Psi\big(\cancelto{x}{F(0,x,y)},\cancelto{y}{G(0,x,y)}\big)$$

$$+\big[\Psi_x\big(\cancelto{x}{F(0,x,y)},\cancelto{y}{G(0,x,y)}\big)\cdot F_s(0,x,y)+\Psi_y\big(\cancelto{x}{F(0,x,y)},\cancelto{y}{G(0,x,y)}\big)\cdot G_s(0,x,y)\big]s$$

$$+O(s^2) \quad = \quad \psi(0)+\psi'(0)s+O(s^2)$$

# Tangents of Symmetries

**Definition:** $\xi(x,y) = F_s(0,x,y)$ and $\eta(x,y) = G_s(0,x,y)$

[Secretly we have the Lie algebra of our 1-parameter Lie group.]

For all points, $(x,y)$, lying on the solution curve we have $\Psi(x,y) = \psi(0)$.

So we get $\psi(0)+\left[\Psi_x\xi+\Psi_y\eta\right]s+O(s^2) = \psi(0)+\psi'(0)s+O(s^2)$.

Therefore, $\Psi_x\xi + \Psi_y\eta = \psi'(0)$ and finally after normalizing $\psi'(0) = 1$, we get

$$\boxed{\Psi_x\,\xi + \Psi_y\,\eta = 1}$$

Using Cramer's rule, we can now solve the system of equations: $A\Psi_x + B\Psi_y = 0$ and $\xi\Psi_x + \eta\Psi_y = 1$.

$$\Psi_x = \det\begin{bmatrix} 0 & B \\ 1 & \eta \end{bmatrix} / \det\begin{bmatrix} A & B \\ \xi & \eta \end{bmatrix} = \frac{-B}{A\eta - B\xi}$$

$$\Psi_y = \det\begin{bmatrix} A & 0 \\ \xi & 1 \end{bmatrix} / \det\begin{bmatrix} A & B \\ \xi & \eta \end{bmatrix} = \frac{A}{A\eta - B\xi}$$

**Define:** $M = \dfrac{1}{A\eta - B\xi}$ is an integrating factor which makes the Pfaffian form of our original equation exact.

In particular, $M(-Bdx + Ady) = M \cdot 0$ yields $\Psi_x dx + \Psi_y dy = 0$.

Therefore, we can solve our original equation:

$$\Psi(x, y) = \text{``}\int \Psi_x dx + \Psi_y dy\text{''} = \text{Constant.}$$

# Reality Sets In

Unfortunately, there is no known way of finding symmetries of a random differential equation from the equation itself. If we could, we could solve all ODEs. In practice, one guesses a possible form that the symmetry could take and then sees if that works.

However, given a symmetry it is possible to determine what the most general ODE with that symmetry looks like. This has been done for some simple symmetries on the next slide.

# Some first-order ODEs and Symmetries

| Equation | $\xi$ | $\eta$ | Equation | $\xi$ | $\eta$ |
|---|---|---|---|---|---|
| $y' = F[y]$ | 1 | 0 | $y' = \frac{y}{x} + xF[\frac{y}{x}]$ | 1 | $\frac{y}{x}$ |
| $y' = F[x]$ | 0 | 1 | $xy' = y + F[\frac{y}{x}]$ | $x^2$ | $xy$ |
| $y' = F[ax + by]$ | $b$ | $-a$ | $y' = \frac{y}{F[\frac{y}{x}]}$ | $xy$ | $y^2$ |
| $y' = \frac{(y+x)F[x^2+y^2]}{(x-y)F[x^2+y^2]}$ | $y$ | $-x$ | $y' = \frac{y}{x+F[y]}$ | $y$ | 0 |
| $y' = F[\frac{y}{x}]$ | $x$ | $y$ | $xy' = y + F[y]$ | 0 | $x$ |
| $y' = x^{k-1}F[\frac{y}{x^k}]$ | $x$ | $ky$ | $xy' = \frac{y}{\ln[x]+F[y]}$ | $xy$ | 0 |
| $xy' = F[xe^{-y}]$ | $x$ | 1 | $xy' = y(\ln[x] + F[x])$ | 0 | $xy$ |
| $y' = yF[ye^{-x}]$ | 1 | $y$ | $y' = yF[x]$ | 0 | $y$ |

where $F$ is an arbitrary function.

# Example 1:

Consider the equation $y' = \frac{xy}{x^2 + y}$.

This equation can be rewritten as $y' = \frac{y}{x + \frac{y}{x}}$.

Looking at the Table from Brian Cantwell's text, we get that $\xi = xy$ and $\eta = y^2$ are symmetry tangents.

Therefore, $M(x, y) = \frac{1}{A\eta - B\xi} = \frac{1}{1 \cdot y^2 - \frac{y}{x + \frac{y}{x}} \cdot xy} = \frac{x^2 + y}{y^3}$ will work as an integrating factor.

The Pfaffian form the equation: $-B\,dx + A\,dy = 0$ becomes $-MB\,dx + MA\,dy = 0$ which after some algebra simplifies to

$$\frac{-x^2}{y(x^2 + y^2)}\,dx + \left(\frac{-x^3}{y^2(x^2 + y^2)} - \frac{1}{y^2}\right)\,dy = 0$$

# Example 1:

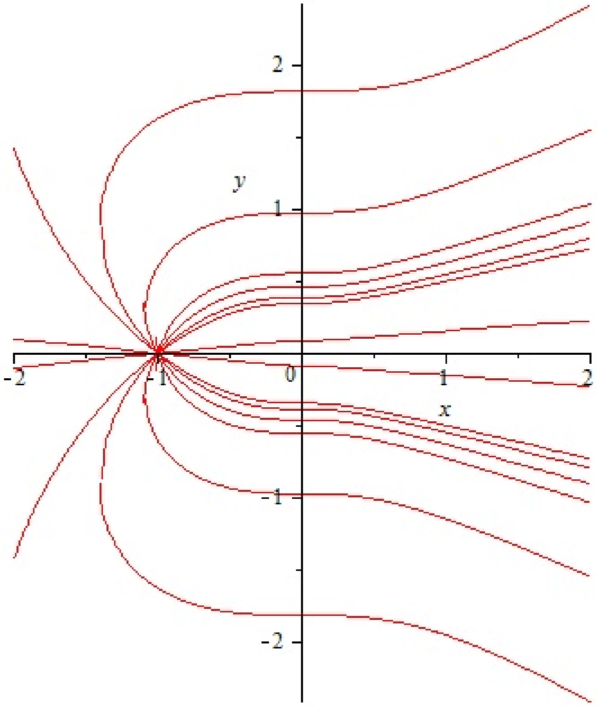
Using standard integrating techniques (i.e. partial fractions), we get:

$$\int \frac{-x^2}{y(x^2+y^2)}\,dx = -\frac{x}{y} + \text{arctan}\left(\frac{x}{y}\right) + C_1(y)$$

$$\int \left(\frac{-x^3}{y^2(x^2+y^2)} - \frac{1}{y^2}\right)\,dy = -\frac{1}{y} - \frac{x}{y} + \text{arctan}\left(\frac{x}{y}\right) + C_2(x)$$

Therefore, the solution is $-\frac{1}{y} - \frac{x}{y} + \text{arctan}\left(\frac{x}{y}\right) = C$.

# Example 1:



Maple can solve this equation as well:

```
--- Trying Lie symmetry methods, 1st order ---
-> Computing symmetries using: way = 2
```

$$\left[0, \frac{x^2y^2+y^4}{x^3+y^2+x^2}\right]$$

```
<- successful computation of symmetries.

trying an integrating factor

                 from the invariance group

<- integrating factor successful
```

# Example 2:

Consider the equation $y' = \frac{y}{x} + x\left(1 + \frac{y^2}{x^2}\right)$.

Looking at the Table from Brian Cantwell's text, we get that $\xi = 1$ and $\eta = y/x$ are symmetry tangents.

Therefore, $M(x,y) = \frac{1}{A\eta - B\xi} = \frac{1}{1 \cdot \frac{y}{x} - \left(\frac{y}{x} + + x\left(1 + \frac{y^2}{x^2}\right)\right) \cdot 1} = -\frac{x}{x^2 + y^2}$ will work as an integrating factor.

The Pfaffian form the equation: $-B\,dx + A\,dy = 0$ becomes $-MB\,dx + MA\,dy = 0$ which after some algebra simplifies to

$$\frac{x - \ln(y)}{x - \ln(y) + 1}\,dx + \frac{1}{x - \ln(y) + 1} \cdot \frac{1}{y}\,dy = 0$$

# Example 2:

After a quick bit of algebra (essentially polynomial division) and a simple $u$-substitution ($u = x - \ln(y) + 1$), we get:

$$\int \frac{x - \ln(y) + 1 - 1}{x - \ln(y) + 1}\, dx = \int 1 - \frac{1}{x - \ln(y) + 1}\, dx$$

$$= x - \ln(x - \ln(y) + 1) + C_1(y)$$

$$\int \frac{1}{x - \ln(y) + 1} \cdot \frac{1}{y}\, dy = -\ln(x - \ln(y) + 1) + C_2(x)$$

Therefore, the solution is $x - \ln(x - \ln(y) + 1) = C$.

# Example 2:

Maple can solve this equation as well:

```
looking for linear symmetries 1st order,
trying the canonical coordinates of the invariance group
-> Computing canonical coordinates for the symmetry [1, y]
-> Calling odsolve with the ODE diff(y(x) x) = y(x) y(x)
      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
-> Computing canonical coordinates for the symmetry [0, (x+1-ln(y))*y]
<- 1st order, canonical coordinates successful
```

# References:

- *Symmetry Methods for Differential Equations: A Beginner's Guide* by Peter E. Hydon, Cambridge Texts in Applied Mathematics, 2000.

- *Introduction to Symmetry Analysis* by Brian J. Cantwell, Cambridge Texts in Applied Mathematics, 2002.

## Thank You!

A big thanks to Dr. T, Dr. Palmer, and Dr. Cook!