Classical Rings

John Adkins $\Leftrightarrow$ adkinsjl1@appstate.edu
Stephen Graham $\Leftrightarrow$ grahamsp@appstate.edu
Michael Kelley $\Leftrightarrow$ kelleyma1@appstate.edu

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Definition: A **ring** is a non-empty set R equipped with two binary operations denoted $a+b$ (addition) and ab (multiplication) such that the following axioms hold...

Associativity For all $a, b, c \in R$, $(a+b)+c = a+(b+c)$

Identity There is some $0 \in R$ such that $a+0 = a = 0+a$ for all $a \in R$

Inverses For each $a \in R$ there exists $-a \in R$ such that

$$a+(-a) = 0 = (-a)+a$$

Commutativity For all $a, b \in R$, $a+b = b+a$

Associativity For all $a, b, c \in R$, $(ab)c = a(bc)$

Identity There is some $1 \in R$ such that $a1 = a = 1a$ for all $a \in R$

Distributive For all $a, b, c \in R$, $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$

If in addition,

Commutativity For all $a, b \in R$, $ab = ba$

then R is called a **commutative ring**.

Examples of Rings

- $\mathbb{Z}_{12} = \{0, 1, 2, \ldots, 11\}$ where for example $8 + 7 = 15 = 3$ and $5 \cdot 4 = 20 = 8$. In this ring we add, subtract, and multiply integers as usual keeping in mind the $12 = 0$ (just like a clock).

- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is a commutative ring.

- $\mathbb{R}$ (real numbers) is a commutative ring.

- $\mathbb{R}^{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}$ is a non-commutative ring.

Definition: An element x in a ring R is called a unit if it has a multiplicative inverse. This means there is some $x^{-1} \in R$ such that $xx^{-1} = 1 = x^{-1}x$.

Examples:

- $1 \cdot 1 = 1$, $5 \cdot 5 = 25 = 1$, $7 \cdot 7 = 49 = 1$, $11 \cdot 11 = 121 = 1$ thus $1, 5, 7$, and 11 are units in $\mathbb{Z}_{12}$.

- $-1, 1 \in \mathbb{Z}$ are units. No other element is. For example, 2^{-1} should be $1/2$ (which is not an integer).

- Every non-zero real number is a unit in $\mathbb{R}$.

- A matrix in $\mathbb{R}^{2\times2}$ is a unit if and only if it has a non-zero determinant.

Definition: Let a and b be non-zero elements of a ring R. If $ab = 0$, then a and b are zero divisors.

Examples:

- $2 \cdot 6 = 12 = 0$, $3 \cdot 4 = 12 = 0$, $8 \cdot 3 = 24 = 0$, $9 \cdot 4 = 36 = 0$, $10 \cdot 6 = 60 = 0$ thus $2, 3, 4, 6, 8, 9,$ and 10 are zero divisors in $\mathbb{Z}_{12}$.

- $\mathbb{Z}$ and $\mathbb{R}$ have no zero divisors. If $ab = 0$ then either $a = 0$ or $b = 0$ in $\mathbb{Z}$ and $\mathbb{R}$.

- A non-zero matrix in $\mathbb{R}^{2\times 2}$ is a zero divisor if and only if its determinant is zero. For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Definition: A ring R is called classical if every element of R is either zero, a zero divisor, or a unit.

Examples:

- $\mathbb{Z}_{12}$ is classical. $\mathbb{Z}_{12} = \{0\} \cup \{1, 3, 5, 7, 11\} \cup \{2, 3, 4, 6, 8, 9, 10\}$
- $\mathbb{Z}$ is not classical. $\pm 2, \pm 3, \ldots$ are non-zero, non-unit, non-zero divisors!
- $\mathbb{R}$ is classical. Recall, in $\mathbb{R}$ all non-zero elements are units.
- $\mathbb{R}^{2\times 2}$ is classical. A non-zero matrix is either a zero divisor or a unit depending on whether its determinant is zero or not.

Our Question:

When is a ring classical?

$\boxed{\text{Theorem: All finite rings are classical.}}$

Sketch of Proof:

Let R be a ring and for $0 \neq r \in R$ define $L_r : R \to R$ by $L_r(x) = rx$.

- r is a unit $\implies L_r$ is onto.
- r is *not a zero divisor* $\implies L_r$ is one-to-one.
- Maps from finite sets to themselves are one-to-one $\iff$ onto.

Examples:

- $\mathbb{Z}_{12}$ or more generally $\mathbb{Z}_n$.
- $(\mathbb{Z}_{12})^{2\times 2}$ or more generally square matrices over any finite ring.

Theorem: Let R be a commutative classical ring, then $R^{n\times n}$ is classical.

Sketch of Proof:

- McCoy's Theorem implies that matrices are zero divisors if their determinants are zero divisors.
- A formula which follows from Cramer's rule implies that matrices whose determinants are units are themselves invertible.
- If the base ring is classical, every determinant is either zero, a zero divisor, or a unit.

Examples:

- $\mathbb{R}^{2\times 2}$ or more generally square matrices over any field (such as $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$).
- $\mathbb{Z}^{2\times 2}$ is not classical since $\mathbb{Z}$ isn't.

Other Results:

- Every commutative ring can be embedded in a classical ring (its total ring of quotients). For example: $\mathbb{Z}$ can be completed to $\mathbb{Q}$.
- Proper quotients of principal ideal domains are classical.
- Artinian rings are classical.
- Rings of Krull dimension 0 are classical.

Open Question:

Artinian and Krull dimension 0 are sort of "finiteness" conditions which imply being classical. In general, what kind of finiteness condition is equivalent to being classical?

Any Questions?