Fractional Derivatives

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What is a

“Fractional Derivative?”

A comprehensive definition of the Gamma function is provided by the Euler limit:

$$\Gamma(x) \equiv \lim_{N\to\infty} \left[\frac{N! \cdot N^x}{x(x+1)(x+2)\cdots(x+N)}\right]$$

The integral transform definition:

$$\Gamma(x) \equiv \int_0^\infty y^{x-1} e^{-y}\, dy \qquad x > 0$$

The most important property of the gamma function:

$$\Gamma(x+1) = x\Gamma(x)$$

It is easy to show:

$$\Gamma(1) = 1$$

Together these imply (for positive integers):

$$\Gamma(n+1) = n!$$

Universal definition:

$$ {}_aD^q f(x) \equiv \lim_{N\to\infty} \left\{ \frac{\left[\frac{x-a}{N}\right]^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(x - j\left[\frac{x-a}{N}\right]\right) \right\} $$

The more practical integral transform definition only works for fractional integration ($q < 0$):

$$ {}_aD^q f(x) = \frac{1}{\Gamma(-q)} \int_a^x (x-s)^{-q-1} f(s)\, ds $$

Computing in Maple

Compute ${}_0D^{1/2}[f(t)]$ where $f(t) = t^2$

$${}_0D_t^{-\frac{1}{2}}(t^2) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{\frac{1}{2}-1} s^2\, ds$$

$$= \frac{1}{\sqrt{\pi}}\int_0^t \frac{s^2}{\sqrt{t-s}}\ ds \qquad = \frac{1}{\sqrt{\pi}}\int_0^t \frac{(t-v)^2}{\sqrt{v}}\ dv\ \ [v = t-s,\ dv = -ds]$$

$$= \frac{1}{\sqrt{\pi}}\int_0^t v^{-\frac{1}{2}}\ t^2 - 2\ t\ v^{\frac{1}{2}} + v^{\frac{1}{2}} + v^{\frac{3}{2}}\ dv$$

$$= \frac{1}{\sqrt{\pi}}\left[\frac{v^{\frac{1}{2}}\ t^2}{\frac{1}{2}} - 2\ t\frac{v^{\frac{3}{2}}}{\frac{3}{2}} + \frac{v^{\frac{5}{2}}}{\frac{5}{3}}\right]_0^t = \frac{16}{15\ \sqrt{\pi}}\ t^{\frac{5}{2}}$$

$${}_0D_t^{1/2}[t^2] = \frac{d}{dt}\left[{}_0D_t^{-1/2}[t^2]\right] = \frac{d}{dt}\left[\frac{16}{15\sqrt{\pi}}t^{5/2}\right] = \frac{8}{3\ \sqrt{\pi}}\ t^{\frac{3}{2}}$$

Applications:

We begin with Bessel's equation used in connection with the vibrations of a circular drumhead

$$x^2 \frac{d^2w}{dx^2} + \frac{dw}{dx} + \left[x - \frac{v^2}{4}t\right] w = 0$$

The diffusion equation is a general equation used in modeling thermodynamics, atmospheric pollutants, and surface electroreduction.

General Diffusion Equation

$$\frac{d}{dt}F(\xi, \eta, \zeta, t) = k\nabla^2 F(\xi, \eta, \zeta, t)$$

With some manipulation and under symmetry conditions electrochemical diffusion can be modeled using the following diffusion equation

A Special Case:

$$\frac{d}{dr}F(r,t) = \frac{-1}{\sqrt{k}}\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}}[F(r,t) - F_0] - \frac{F(r,t) - F_0}{r+R}$$

Reference

- M. Kleinz and T. J. Osler , A Child's Garden of Fractional Derivatives

- K. S. Miller, Derivatives of Noninteger Order

- K. B. Oldham and J. Spanier, The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order

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