MA 792M: Special Topics in Lie Algebras

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Abstract

These are notes from a presentation for MA 792M (Representation Theory for Lie Algebras) taught by Professor Kailash Misra. The goal is to present a construction of G_2. In the first section I will present a summary of basic facts about G_2. In the second section I construct the root system of G_2 in R^3 using the root system of B_3 (which is the 7x7 orthoginal matrix algebra). In the third section I will construct G_2 itself (as a subalgebra of $o(7, C)$). In the final section I will *sketch* a construction of G_2 as the derivation algebra of the octonians. My primary source was *Introduction to Lie Algebras and Representation Theory* by James E. Humphreys pages 102-106.

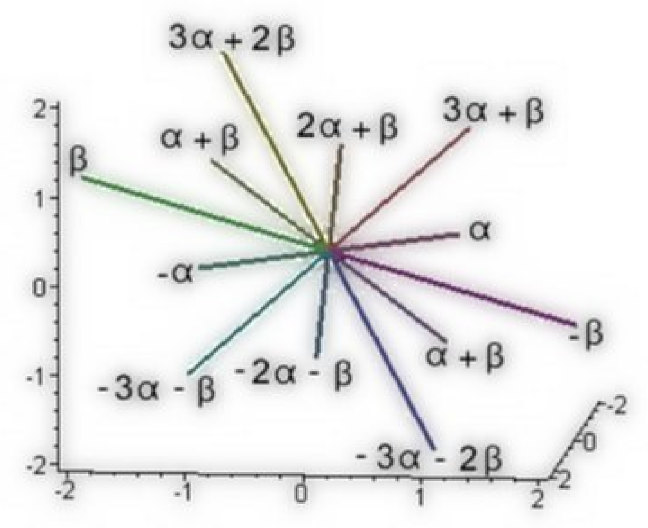
1 Preliminaries

Starting with the Dynkin diagram, we see that G_2's Cartan matrix is:

$$A = (a_{ij}) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Let us label our simple roots as α and β. (Let $\pi = \{\alpha, \beta\}$ be our set of simple roots). Using the algorithm outlined in Humphreys it is easy to find all the positive roots (hence all the roots). After calculating α and β root strings, we find that $\Phi^+ = \{\alpha,\, \beta,\, \alpha+\beta,\, 2\alpha+\beta,\, 3\alpha+\beta,\, 3\alpha+2\beta\}$ are all the positive roots thus there are 12 roots all together:

$$\Phi = \{\pm\alpha,\, \pm\beta,\, \pm(\alpha+\beta),\, \pm(2\alpha+\beta),\, \pm(3\alpha+\beta),\, \pm(3\alpha+2\beta)\}$$



The Root System of G_2

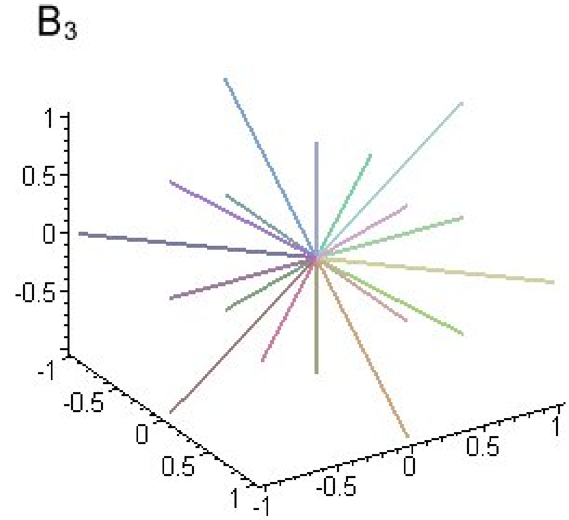
Note: The long roots form a root system of type A_2 (this we be important later).

Thus a Lie algebra of type G_2 must be $2 + 12 = 14$-dimensional. G_2's Weyl group is generated by the two reflections: $r_\alpha(x) = x - < x, \alpha > \alpha$ and $r_\beta(x) = x - < x, \beta > \beta$ where $x \,\epsilon\, E$ (x is in the Euclidean space spanned by α and β). Now using a proposition from class (noting that $a_{12}a_{21} = 3$) we see that the order of $r_\alpha r_\beta$ is 6. Thus the Weyl group is isomorphic to D_6 (dihedral group of order 12).

2 The Root System

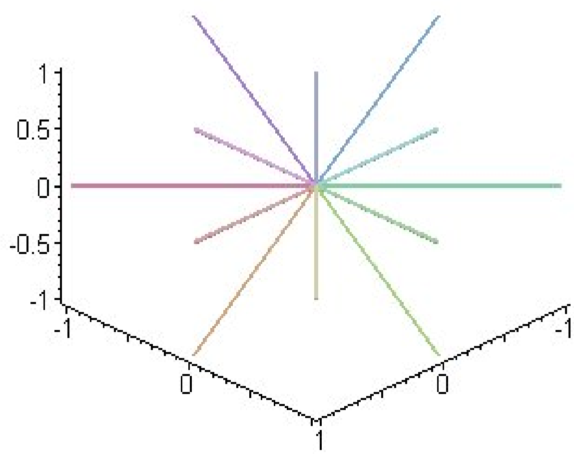
Using representation theory one can show that G_2 has an irreducible faithful representation by 7x7 matricies (and in fact this is the smallest irreducible faithful representation).

Consider $o(7, C)$. Let us construct the root system in R^3 (this is constructed in Humphreys page 64). Let (e_1, e_2, e_3) be the standard basis for R^3. Then $\{e_1 - e_2,\, e_2 - e_3,\, e_3\}$ is also a basis for R^3. For the short roots we have $\pm e_1, \pm e_2, \pm e_3$. For the long roots we have $\pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3),\, \pm(e_1 + e_2), \pm(e_1 + e_3), \pm(e_2 + e_3)$.



The Root System of $o(7, C)$

Now rotating and viewing this root system from the right angle we see the root system for G_2



The Root System of $o(7, C)$ (different view)

Now to construct the root system of G_2. We want to project the roots of B_3 onto the plane $E = \{(x, y, z) \,|\, x+y+z = 0\}$ (the plane orthoginal to $e_1+e_2+e_3$).

Projecting to this plane we get:

$$\pm\{e_1-e_2,\ e_1-e_3,\ e_2-e_3,\ \frac{2}{3}e_1-\frac{1}{3}e_2-\frac{1}{3}e_3,\ -\frac{1}{3}e_1+\frac{2}{3}e_2-\frac{1}{3}e_3,\ -\frac{1}{3}e_1-\frac{1}{3}e_2+\frac{2}{3}e_3\}$$

which happens to be the root system of G_2.

Note: The long roots (of G_2) form a root system of type A_2 and also these roots are precisely the short roots of B_3.

3 Constructing G_2 (as a subalgebra of $o(7, C)$)

Let's start with the classical Lie algebra $L_0 = o(7, C)$. The standard basis for L_0 in terms of matrix units $E_{i,j}$ (7 x 7 complex matrix with 1 in the i, j^{th} position everything else zero):

$g_{i,-j} = E_{i+1,j+1} - E_{j+4,i+4}$ for $1 \leq i \neq j \leq 3$ (6 matrices)

$d_i = E_{i+1,i+1} - E_{i+4,i+4}$ for $1 \leq i \leq 3$ (3 matrices)

$E_{1,i+4} - E_{i+1,1}$ and $E_{1,i+1} - E_{i+4,1}$ for $1 \leq i \leq 3$ (3 + 3 matrices)

$E_{i+1,j+4} - E_{j+1,i+4}$ and $E_{i+4,j+1} - E_{j+4,i+1}$ for $1 \leq i < j \leq 3$ (3 + 3 matrices)

A Cartan subalgebra (CSA i.e. maximal toral subalgebra) of L_0 is $H_0 = span\{d_1, d_2, d_3\}$. For the CSA of L (our type G_2 Lie algebra) we will use $H = span\{d_1 - d_2, d_2 - d_3\} = \{\sum a_i d_i \,|\, \sum a_i = 0\}$. Let L be the span of H together with $g_{i,-j}$ where $1 \leq i \neq j \leq 3$ (these correspond to the 6 long roots which form a type A_2 root system).

Now when we formed the root system of G_2 from that of B_3 we had to project onto a plane and collapse 3 pairs of roots from B_3 into the 3 short roots of G_2. So we need a combination of elements from these pairs of root spaces (after some work) the correct choices are:

$g_1 = -g_{-1}^t = \sqrt{2}(E_{1,2} - E_{5,1}) - (E_{3,7} - E_{4,6})$

$g_2 = -g_{-2}^t = \sqrt{2}(E_{1,3} - E_{6,1}) + (E_{2,7} - E_{4,5})$

$g_3 = -g_{-3}^t = \sqrt{2}(E_{1,4} - E_{7,1}) - (E_{2,6} - E_{3,5})$

OR $g_i = -g_{-i}^t = \sqrt{2}(E_{1,i+1} - E_{i+4,1}) - \delta_{i1}(E_{3,7} - E_{4,6}) + \delta_{i2}(E_{2,7} - E_{4,5}) - \delta_{i3}(E_{2,6} - E_{3,5})$

Thus L is a subspace of L_0 of dimension 14. First we need to show that L is closed under the commutator bracket. Recall that $[E_{i,j}, E_{k,l}] = \delta_{jk}E_{i,l} - \delta_{li}E_{k,j}$ where $\delta_{ij} = \left\{ \begin{array}{cc} 1 & i = j \\ 0 & i \neq j \end{array} \right\}$ is the Kronecker delta. We need the following relations $1 \leq i,\ j,\ k,\ l \leq 3$:

1. $[g_{i,-j},\ g_{j,-i}] = d_i - d_j$ for $i \neq j$
   $[g_{i,-j},\ g_{k,-l}] = \delta_{jk} g_{i,-l} - \delta_{il} g_{k,-j}$ for $i \neq j$, $k \neq l$, and $(i,j) \neq (l,k)$
2. $[g_i,\ g_{-i}] = 3d_i - (d_1 + d_2 + d_3)$
3. $[g_{i,-j},\ g_k] = -\delta_{ik} g_j$
   $[g_{i,-j},\ g_{-k}] = \delta_{jk} g_{-i}$
4. $[g_i,\ g_{-j}] = 3g_{j,-i}$ for $i \neq j$
5. $[g_i,\ g_j] = \pm 2g_{-k}$ for $i,\ j,\ k$ distinct
   $[g_{-i},\ g_{-j}] = \pm 2g_k$ for $i,\ j,\ k$ distinct (for the exact signs see the following proof)

(1.)
$[g_{i,-j}, g_{k,-l}] = [E_{i+1,j+1} - E_{j+4,i+4}, E_{k+1,l+1} - E_{l+4,k+4}] = [E_{i+1,j+1}, E_{k+1,l+1}] - [E_{i+1,j+1}, E_{l+4,k+4}] - [E_{j+4,i+4}, E_{k+1,l+1}] + [E_{j+4,i+4}, E_{l+4,k41}] = \delta_{jk} E_{i+1,l+1} - \delta_{li} E_{k+1,j+1} - 0 - 0 - \delta_{jk} E_{l+4,i+4} + \delta_{li} E_{k+4,j+4}$

$[g_{i,-j},\ g_{j,-i}] = \delta_{jk} g_{i,-l} - \delta_{li} g_{k,-j}$ for $(i,j) \neq (l,k)$ and
$[g_{i,-j}, g_{j,-i}] = E_{i+1,i+1} - E_{j+1,j+1} - E_{i+4,i+4} + E_{j+4,j+4} = d_i - d_j$ for $(i,j) = (l,k)$

(2.)
$[g_1, g_{-1}] = [\sqrt{2}(E_{1,2} - E_{5,1}) - (E_{3,7} - E_{4,6}), -\sqrt{2}(E_{2,1} - E_{1,5}) + (E_{7,3} - E_{6,4})] = -2[E_{1,2},\ E_{2,1}] - 2[E_{5,1},\ E_{1,5}] - [E_{3,7},\ E_{7,3}] - [E_{4,6},\ E_{6,4}] = -2E_{1,1} + 2E_{2,2} - 2E_{5,5} + 2E_{1,1} - E_{3,3} + E_{7,7} - E_{4,4} + E_{6,6} = 2d_1 - d_2 - d_3 = 3d_1 - (d_1 + d_2 + d_3)$

$[g_2, g_{-2}] = [\sqrt{2}(E_{1,3} - E_{6,1}) + (E_{2,7} - E_{4,5}), -\sqrt{2}(E_{3,1} - E_{1,6}) - (E_{7,2} - E_{5,4})] = -2[E_{1,3},\ E_{3,1}] - 2[E_{6,1},\ E_{1,6}] - [E_{2,7},\ E_{7,2}] - [E_{4,5},\ E_{5,4}] = -2E_{1,1} + 2E_{3,3} - 2E_{6,6} + 2E_{1,1} - E_{2,2} + E_{7,7} - E_{4,4} + E_{5,5} = 2d_2 - d_1 - d_3 = 3d_2 - (d_1 + d_2 + d_3)$

$[g_3, g_{-3}] = [\sqrt{2}(E_{1,4} - E_{7,1}) - (E_{2,6} - E_{3,5}), -\sqrt{2}(E_{4,1} - E_{1,7}) + (E_{6,2} - E_{5,3})] = -2[E_{1,4},\ E_{4,1}] - 2[E_{7,1},\ E_{1,7}] - [E_{2,6},\ E_{6,2}] - [E_{3,5},\ E_{5,3}] = -2E_{1,1} + 2E_{4,4} - 2E_{7,7} + 2E_{1,1} - E_{2,2} + E_{6,6} - E_{3,3} + E_{5,5} = 2d_3 - d_1 - d_2 = 3d_3 - (d_1 + d_2 + d_3)$

(3.)
$[g_{i,-j},\ g_k] = [E_{i+1,j+1} - E_{j+4,i+4},\ \sqrt{2}(E_{1,k+1} - E_{k+4,1}) - \delta_{k1}(E_{3,7} - E_{4,6}) + \delta_{k2}(E_{2,7} - E_{4,5}) - \delta_{k3}(E_{2,6} - E_{3,5})] = [E_{i+1,j+1} - E_{j+4,i+4}, \sqrt{2}(E_{1,k+1} - E_{k+4,1})] = -\delta_{ik}\sqrt{2}E_{1,j+1} + \delta_{ik}\sqrt{2}E_{j+4,1}$

$(k = 1)$ $[E_{i+1,j+1} - E_{j+4,i+4},\ -(E_{3,7} - E_{4,6})] = -\delta_{j2} E_{i+1,7} + \delta_{j3} E_{i+1,6} - \delta_{j3} E_{3,i+4} + \delta_{j2} E_{4,i+4}$
$(k = 2)$ $[E_{i+1,j+1} - E_{j+4,i+4}, (E_{2,7} - E_{4,5})] = \delta_{j1} E_{i+1,7} - \delta_{j3} E_{i+1,5} + \delta_{j3} E_{2,i+4} - \delta_{j1} E_{4,i+4}$
$(k = 3)$ $[E_{i+1,j+1} - E_{j+4,i+4},\ -(E_{2,6} - E_{3,5})] = -\delta_{j1} E_{i+1,6} + \delta_{j2} E_{i+1,5} - \delta_{j2} E_{2,i+4} + \delta_{j1} E_{3,i+4}$

$[g_{i,-j},\ g_k] = -\delta_{ik}\sqrt{2}(E_{1,j+1} - E_{j+4,1}) + \delta_{1k}(\delta_{j2}(E_{4,i+4} - E_{i+1,7}) + \delta_{j3}(E_{i+1,6} - E_{3,i+4})) - \delta_{2k}(\delta_{j1}(E_{4,i+4} - E_{i+1,7}) + \delta_{j3}(E_{i+1,5} - E_{2,i+4})) + \delta_{3k}(\delta_{j1}(E_{3,i+4} - E_{i+1,6}) + \delta_{j2}(E_{i+1,5} - E_{2,i+4})) = -\delta_{ik}g_j$

$[g_{i,-j},\ g_{-k}] = [E_{i+1,j+1} - E_{j+4,i+4},\ \sqrt{2}(E_{1,k+4} - E_{k+1,1}) + \delta_{k1}(E_{7,3} - E_{6,4}) - \delta_{k2}(E_{7,2} - E_{5,4}) + \delta_{k3}(E_{6,2} - E_{5,3})] = [E_{i+1,j+1} - E_{j+4,i+4}, \sqrt{2}(E_{1,k+4} - E_{k+1,1})] = -\delta_{jk}\sqrt{2}E_{i+1,1} + \delta_{jk}\sqrt{2}E_{1,i+4}$

$(k=1)$ $[E_{i+1,j+1} - E_{j+4,i+4}, (E_{7,3} - E_{6,4})] = -\delta_{i2}E_{7,j+1} + \delta_{i3}E_{6,j+1} - \delta_{i3}E_{j+4,3} + \delta_{i2}E_{j+4,4}$
$(k=2)$ $[E_{i+1,j+1} - E_{j+4,i+4}, -(E_{7,2} - E_{5,4})] = \delta_{i1}E_{7,j+1} - \delta_{i3}E_{5,j+1} + \delta_{i3}E_{j+4,2} - \delta_{i1}E_{j+4,4}$
$(k=3)$ $[E_{i+1,j+1} - E_{j+4,i+4}, (E_{6,2} - E_{5,3})] = -\delta_{i1}E_{6,j+1} + \delta_{i2}E_{5,j+1} - \delta_{i2}E_{j+4,2} + \delta_{i1}E_{j+4,3}$

$[g_{i,-j},\ g_{-k}] = \delta_{jk}\sqrt{2}(E_{1,i+4} - E_{i+1,1}) - \delta_{1k}(\delta_{i2}(E_{7,j+1} - E_{j+4,4}) + \delta_{i3}(E_{j+4,3} - E_{6,j+1})) + \delta_{2k}(\delta_{i1}(E_{7,j+1} - E_{j+4,4}) + \delta_{i3}(E_{j+4,2} - E_{5,j+1})) - \delta_{3k}(\delta_{i1}(E_{6,j+1} - E_{j+4,3}) + \delta_{i2}(E_{j+4,2} - E_{5,i+4})) = \delta_{jk}g_{-i}$

(4.)
$[g_1,\ g_{-2}] = [\sqrt{2}(E_{1,2} - E_{5,1}) - (E_{3,7} - E_{4,6}),\ -\sqrt{2}(E_{3,1} - E_{1,6}) - (E_{7,2} - E_{5,4})] = -2[E_{1,2},\ E_{3,1}] - 2[E_{5,1},\ E_{1,6}] + [E_{3,7},\ E_{7,2}] + [E_{4,6},\ E_{5,4}] = 2E_{3,2} - 2E_{5,6} + E_{3,2} - E_{5,6} = 3g_{2,-1}$

$[g_1,\ g_{-3}] = [\sqrt{2}(E_{1,2} - E_{5,1}) - (E_{3,7} - E_{4,6}),\ -\sqrt{2}(E_{4,1} - E_{1,7}) + (E_{6,2} - E_{5,3})] = -2[E_{1,2},\ E_{4,1}] - 2[E_{5,1},\ E_{1,7}] + [E_{4,6},\ E_{6,2}] + [E_{3,7},\ E_{5,3}] = 2E_{4,2} - 2E_{5,7} + E_{4,2} - E_{5,7} = 3g_{3,-1}$

$[g_2,\ g_{-3}] = [\sqrt{2}(E_{1,3} - E_{6,1}) + (E_{2,7} - E_{4,5}),\ -\sqrt{2}(E_{4,1} - E_{1,7}) + (E_{6,2} - E_{5,3})] = -2[E_{1,3},\ E_{4,1}] - 2[E_{6,1},\ E_{1,7}] + [E_{2,7},\ E_{6,2}] + [E_{4,5},\ E_{5,3}] = 2E_{4,3} - 2E_{6,7} - E_{6,7} + E_{4,3} = 3g_{2,-3}$

Now note that $[g_i,\ g_{-j}]^t = (g_i g_{-j})^t - (g_{-j}g_i)^t = g_{-j}^t g_i^t - g_i^t g_{-j}^t = (-g_j)(-g_{-i}) - (-g_{-i})(-g_j) = g_j g_{-i} - g_{-i}g_j = [g_j,\ g_{-i}]$. Thus we have that: $[g_2,\ g_{-1}] = [g_1,\ g_{-2}]^t = 3g_{2,-1}^t = 3g_{1,-2}$, $[g_3,\ g_{-1}] = [g_1,\ g_{-3}]^t = 3g_{3,-1}^t = 3g_{1,-3}$, and $[g_3,\ g_{-2}] = [g_2,\ g_{-3}]^t = 3g_{3,-2}^t = 3g_{2,-3}$

(5.)
$[g_1,\ g_2] = [\sqrt{2}(E_{1,2} - E_{5,1}) - (E_{3,7} - E_{4,6}),\ \sqrt{2}(E_{1,3} - E_{6,1}) + (E_{2,7} - E_{4,5})] = -2[E_{1,2},\ E_{6,1}] - 2[E_{5,1},\ E_{1,3}] + \sqrt{2}[E_{1,2},\ E_{2,7}] + \sqrt{2}[E_{5,1},\ E_{4,5}] - \sqrt{2}[E_{3,7},\ E_{1,3}] - \sqrt{2}[E_{4,6},\ E_{6,1}] = 2E_{6,2} - 2E_{5,3} + \sqrt{2}E_{1,7} - \sqrt{2}E_{4,1} + \sqrt{2}E_{1,7} - \sqrt{2}E_{4,1} = 2g_{-3}$

$[g_1,\ g_3] = [\sqrt{2}(E_{1,2} - E_{5,1}) - (E_{3,7} - E_{4,6}),\ \sqrt{2}(E_{1,4} - E_{7,1}) - (E_{2,6} - E_{3,5})] = -2[E_{1,2},\ E_{7,1}] - 2[E_{5,1},\ E_{1,4}] - \sqrt{2}[E_{1,2},\ E_{2,6}] + \sqrt{2}[E_{5,1},\ E_{3,5}] - \sqrt{2}[E_{3,7},\ E_{7,1}] - \sqrt{2}[E_{4,6},\ E_{1,4}] = 2E_{7,2} - 2E_{5,4} - \sqrt{2}E_{1,6} + \sqrt{2}E_{3,1} - \sqrt{2}E_{1,6} + \sqrt{2}E_{3,1} = -2g_{-2}$

$[g_2,\ g_3] = [\sqrt{2}(E_{1,3} - E_{6,1}) + (E_{2,7} - E_{4,5}),\ \sqrt{2}(E_{1,4} - E_{7,1}) - (E_{2,6} - E_{3,5})] = -2[E_{1,3},\ E_{7,1}] - 2[E_{6,1},\ E_{1,4}] + \sqrt{2}[E_{1,3},\ E_{3,5}] + \sqrt{2}[E_{6,1},\ E_{2,6}] - \sqrt{2}[E_{2,7},\ E_{7,1}] - \sqrt{2}[E_{4,5},\ E_{1,4}] = 2E_{7,3} - 2E_{6,4} + \sqrt{2}E_{1,5} - \sqrt{2}E_{2,1} - \sqrt{2}E_{2,1} + \sqrt{2}E_{1,5} = 2g_{-1}$

Now note that $[g_i,\ g_j]^t = (g_i g_j)^t - (g_j g_i)^t = g_j^t g_i^t - g_i^t g_j^t = (-g_{-j})(-g_{-i}) - (-g_{-i})(-g_{-j}) = g_{-j}g_{-i} - g_{-i}g_{-j} = [g_{-j},\ g_{-i}]$. Thus we have that: $[g_{-1},\ g_{-2}] = [g_2,\ g_1]^t = -2g_{-3}^t = 2g_3$, $[g_{-1},\ g_{-3}] = [g_3,\ g_1]^t = 2g_{-2}^t = -2g_2$, and $[g_{-2},\ g_{-3}] = [g_3,\ g_2]^t = -2g_{-1}^t = 2g_1$

The only relations left to check are those involving $d_1 - d_2$ and $d_2 - d_3$. First note that: $[d_1 - d_2,\ d_2 - d_3] = 0$ because d_i's are diagonal (hence commute).

Now consider $[d_i,\ g_{j,-k}] = [E_{i+1,i+1} - E_{i+4,i+4},\ E_{j+1,k+1} - E_{k+4,j+4}] = [E_{i+1,i+1}, E_{j+1,k+1}] - [E_{i+4,i+4}, E_{j+1,k+1}] - [E_{i+1,i+1}, E_{k+4,j+4}] + [E_{i+4,i+4}, E_{k+4,j+4}] = \delta_{ij}E_{i+1,k+1} - \delta_{ik}E_{j+1,i+1} + \delta_{ik}E_{i+4,j+4} - \delta_{ij}E_{k+4,i+4} = \delta_{ij}g_{i,-k} - \delta_{ik}g_{j,-i}$

Thus we have that $[d_1 - d_2,\ g_{j,-k}] = \delta_{1j}g_{j,-k} - \delta_{1k}g_{j,-k} - \delta_{2j}g_{j,-k} + \delta_{2k}g_{j,-k}$ $= (\delta_{1j} - \delta_{1k} - \delta_{2j} + \delta_{2k})g_{j,-k}$ which is non-zero since j and k are distinct. Likewise $[d_2 - d_3,\ g_{j,-k}] = Cg_{i,-k}$ where C is non-zero. We have only g_i's left to check.

$[d_1 - d_2,\ g_1] = [E_{2,2} - E_{5,5} - E_{3,3} + E_{6,6},\ \sqrt{2}(E_{1,2} - E_{5,1}) - (E_{3,7} - E_{4,6})]$ $= \sqrt{2}[E_{2,2},\ E_{1,2}] + \sqrt{2}[E_{5,5},\ E_{5,1}] + [E_{3,3},\ E_{3,7}] + [E_{6,6},\ E_{4,6}] = -\sqrt{2}E_{1,2} + \sqrt{2}E_{5,1} + E_{3,7} - E_{4,6} = -g_1$

$[d_2 - d_3,\ g_1] = [E_{3,3} - E_{6,6} - E_{4,4} + E_{7,7},\ \sqrt{2}(E_{1,2} - E_{5,1}) - (E_{3,7} - E_{4,6})]$ $= -[E_{3,3}, E_{3,7}] - [E_{6,6}, E_{4,6}] - [E_{4,4}, E_{4,6}] - [E_{7,7}, E_{3,7}] = -E_{3,7} + E_{4,6} - E_{4,6} + E_{3,7}$ $= 0$

$[d_1 - d_2,\ g_2] = [E_{2,2} - E_{5,5} - E_{3,3} + E_{6,6},\ \sqrt{2}(E_{1,3} - E_{6,1}) + (E_{2,7} - E_{4,5})]$ $= [E_{2,2}, E_{2,7}] + [E_{5,5}, E_{4,5}] - \sqrt{2}[E_{3,3}, E_{1,3}] - \sqrt{2}[E_{6,6}, E_{6,1}] = \sqrt{2}E_{1,3} - \sqrt{2}E_{6,1} + E_{2,7} - E_{4,5} = g_2$

$[d_2 - d_3,\ g_2] = [E_{3,3} - E_{6,6} - E_{4,4} + E_{7,7},\ \sqrt{2}(E_{1,3} - E_{6,1}) + (E_{2,7} - E_{4,5})]$ $= \sqrt{2}[E_{3,3},\ E_{1,3}] - \sqrt{2}[E_{6,6},\ E_{6,1}] + [E_{4,4},\ E_{4,5}] + [E_{7,7},\ E_{2,7}] = -\sqrt{2}E_{1,3} + \sqrt{2}E_{6,1} + E_{4,5} - E_{2,7} = -g_2$

$[d_1 - d_2,\ g_3] = [E_{2,2} - E_{5,5} - E_{3,3} + E_{6,6},\ \sqrt{2}(E_{1,4} - E_{7,1}) - (E_{2,6} - E_{3,5})]$ $= -[E_{2,2}, E_{2,6}] - [E_{5,5}, E_{3,5}] - [E_{3,3}, E_{3,5}] - [E_{6,6}, E_{2,6}] = -E_{2,6} + E_{3,5} - E_{3,5} + E_{2,6}$ $= 0$

$[d_2 - d_3,\ g_3] = [E_{3,3} - E_{6,6} - E_{4,4} + E_{7,7},\ \sqrt{2}(E_{1,4} - E_{7,1}) - (E_{2,6} - E_{3,5})]$ $= [E_{3,3}, E_{3,5}] + [E_{6,6}, E_{2,6}] - \sqrt{2}[E_{4,4}, E_{1,4}] - \sqrt{2}[E_{7,7}, E_{7,1}] = \sqrt{2}E_{1,4} - \sqrt{2}E_{7,1} - E_{2,6} + E_{3,5} = g_3$

Because d_i's are diagonal we have that $[d_i,\ g_{-j}]^t = (d_i g_{-j} - g_{-j}d_i)^t = g_{-j}^t d_i^t - d_i^t g_{-j}^t = -g_j d_i + d_i g_j = [d_i,\ g_j]$ thus $[d_1 - d_2,\ g_{-1}] = g_{-1}$, $[d_2 - d_3,\ g_{-1}] = 0$,

$[d_1 - d_2, g_{-2}] = -g_{-2}$, $[d_2 - d_3, g_{-2}] = g_{-2}$, $[d_1 - d_2, g_{-3}] = 0$, and $[d_2 - d_3, g_{-3}] = -g_{-3}$.

Thus we have shown that L is closed under bracket (thus a subalgebra of L_0) and furthermore, we have shown that the elements of L are common eigenvectors of $ad(H)$. We have that $N_L(H) = \{x \,\epsilon\, L \,|\, [x, H] \subseteq H\} = H$ (H is self-normalizing), since all $x \,\epsilon\, L$ which aren't in H are eigenvectors with non-zero eigenvalues for some $h \,\epsilon\, H$. Now $[H, H] = \{0\}$ (H is abelian) thus H is nilpotent, hence a CSA (self-normalizing and nilpotent).

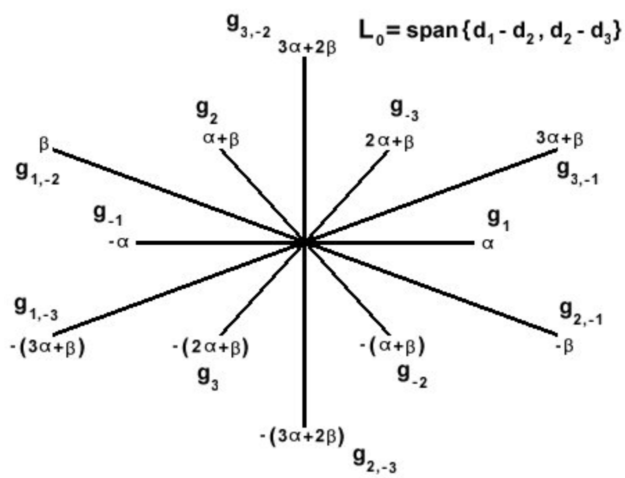
We now have have a 14-dimensional Lie algebra L with a 2-dimensional CSA. Thus if L is semi-simple, by the classification theorem, L must be of type G_2.

Proposition 1 *Let $L \subseteq sl(V)$ (where V is finite dimensional) be a non-zero Li algebra acting irreducibly on V, then L is semi-simple.*

Proof. See Humphreys page 102. ■

Consider L acting on C^7(by regular matrix multiplication). Let $(e_i)_{i=1}^7$ be the standard basis for C^7. $(d_1 - d_2) + 3(d_2 - d_3) = E_{2,2} + 2E_{3,3} - 3E_{4,4} - E_{5,5} - 2E_{6,6} + 3E_{7,7} \,\epsilon\, H$ this matrix has distinct eigenvalues which implies that any non-zero subspace of C^7 invariant under L must contain at least one of the standard basis vectors. Now consider the following:
$g_{-1}e_1 = \sqrt{2}e_2$, $g_{2,-1}e_2 = e_3$, $g_{3,-2}e_3 = e_4$, $g_{-2}e_4 = e_5$, $g_{1,-2}e_5 = -e_6$, and $g_{2,-3}e_6 = -e_7$
Thus if one basis vector is in a subspace then all of the basis vectors are in that subspace. Thus the only invariant subspaces are zero and L itself. Therefore, L acts irreducibly on C^7.

Finally, a quick examination of the basis for L shows that each matrix has trace zero thus $L \subseteq sl(7, C)$. Therefore by the above proposition L must be semi-simple, hence of type G_2.



Root Space Decomposition

4 Constructing G_2 (as the derivation algebra of the octonians)

We now sketch the construction of G_2 as the derivation algebra of the octonians. First we must construct the octonians and review the definition of a derivation.

Let Θ be a 8-dimensional complex vector space whose elements are 2x2 matrices with complex entries down the diagonal and 3-vectors on the off diagonals. That is

$$\Theta = \left\{ \left(\begin{array}{cc} a & v \\ w & b \end{array} \right) \middle| \, a, b \,\epsilon\, C \, and \, v, w \,\epsilon\, C^3 \right\}$$

We give Θ its vector space structure by defining scalar multiplication and vector addition componentwise. As for vector multiplication we take the following:

$$\left(\begin{array}{cc} a & v \\ w & b \end{array} \right) \left(\begin{array}{cc} c & y \\ z & d \end{array} \right) = \left(\begin{array}{cc} ac - v \cdot z & ay + dv + w \times z \\ cw + bz + v \times y & bd - w \cdot y \end{array} \right)$$

where $v \cdot z$ is the standard dot product and $v \times y$ is the standard cross product. Θ is a non-associative algebra called the *Cayley* or *octonian* algebra. Fixing a basis and examining the multiplication table for Θ (see Humphreys page 104-5), we can see that $[\Theta, \Theta] = \Theta_0$ is the span of

$$\left\{ \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \left(\begin{array}{cc} 0 & e_1 \\ 0 & 0 \end{array} \right), \left(\begin{array}{cc} 0 & e_2 \\ 0 & 0 \end{array} \right), \left(\begin{array}{cc} 0 & e_3 \\ 0 & 0 \end{array} \right), \left(\begin{array}{cc} 0 & 0 \\ e_1 & 0 \end{array} \right), \left(\begin{array}{cc} 0 & 0 \\ e_2 & 0 \end{array} \right), \left(\begin{array}{cc} 0 & 0 \\ e_3 & 0 \end{array} \right) \right\}$$

(trace 0 elements). The complement is spanned by $\left\{ \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \right\}$.

To review, a derivation is a linear transformation which satisfies the Leibnitz rule: $\varphi(xy) = \varphi(x)y + x\varphi(y)$ (product rule). Let $L = Der(\Theta)$ (the set of all derivations). It is easy to verify that a linear combination of derivations is still a derivation thus L is a complex vector space. Also one can easily verify that the commutator of two derivations is still a derivation. Thus L is a Lie algebra with commutator bracket.

Consider $\varphi \,\epsilon\, L$ and $x = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$ (the muliplicative identity of the octonians) then $\varphi(x) = \varphi(xx) = \varphi(x)x + x\varphi(x) = 2\varphi(x)$ thus $\varphi(x) = 0$ (so "derivations kill the constants"). Derivations leave Θ_0 invariant, thus L acts faithfully on Θ_0 and trivially on its complement. Let $\phi : L \to gl(7, C)$ be the associated matrix representation of L (using the above basis).

We need to show that L is simple and has a 2-dimensional CSA, but we also need that L is sufficiently large (to rule out A_2, B_2, C_2, and D_2). To do this first consider the following transformations: $\delta(x) \left(\begin{array}{cc} a & v \\ w & b \end{array} \right) = \left(\begin{array}{cc} 0 & x(v) \\ -x^t(w) & 0 \end{array} \right)$ where $x \,\epsilon\, sl_3(C)$. Each $\delta(x)$ is a derivation in L. Thus we have a non-trivial hence faithful representation of $sl_3(C)$. Let $M = \{\delta(x) \,|\, x \,\epsilon\, sl_3(C)\}$. Let H be the image of the diagonal subalgebra of $sl_3(C)$. A simple calculation shows that

H is its own centralizer in L. We know that $Z(M) = \{0\}$ since M is simple thus we also have that $Z(L) = \{0\}$. Therefore L is simple and has a CSA, H, which is 2-dimensional. To show that L must be of type G_2 we still need to show it is bigger than the other rank 2 simple Lie algebras. To do this we note that Θ is an alternative algebra, this is it satifies:

$$x^2y = x(xy)$$

$$yx^2 = (yx)x$$

Let $D_{a,b}$ be a linear transformation defined by:

$$D_{a,b} = [l_a, l_b] + [l_a, r_b] + [r_a, r_b]$$

where $a, b \,\epsilon\, \Theta$ and l_a, r_a denote left and right multiplcation respectively. Then $D_{a,b}$ is a derivation. These provide the remaining derivations needed to show that L is of type G_2. For more details see: *Lie Algebras* by Nathan Jacobson pages 142-145 and *An Introduction to Nonassociative Algebras* by Richard D. Schafer pages 75-90.