Infinity lies at the heart of a large portion of modern mathematics. When studying calculus, one necessarily runs into infinity all over the place. Let's start by asking, "What is infinity?" Essentially infinity (denoted by ∞) is a quantity larger than all finite quantities. Since ∞ is too mind bending to deal with directly most of the time, we tend to approximate ∞ by replacing it with a *really big* number.

Interesting Fact: Did you know there are different "sizes" of infinity? The set of positive integers $\{1, 2, 3, \ldots\}$ is infinite, yet it is not as big as the set of real numbers $\mathbb{R}$. Try googling the terms cardinality and ordinal number and check out the wikipedia articles.

Our goal today is to learn how *fast* certain functions get to ∞. Specifically, we want to know which function gets there the fastest and how this affects graphs.

Experiment: Try plugging bigger and bigger numbers into x^2, x^5, $\sqrt{x}$, 2^x, 3^x, and $\left(\frac{1}{2}\right)^x$. Do all of these expressions head to ∞? Which one(s) get there the fastest?

Notice $(\frac{1}{2})^1 = 0.5$, $(\frac{1}{2})^2 = 0.25$, $(\frac{1}{2})^3 = 0.125, \ldots$ is heading to 0 (not ∞). The other functions head toward ∞. From slowest to fastest: $\sqrt{x}$, x^2, x^5, 2^x, and 3^x.

In general, powers of x (like x^2, x^{15}, $\sqrt{x} = x^{1/2}$, etc.) grow more slowly than exponential functions (like 2^x, 3^x, 10^x) – unless we have an exponential function whose base is 1 ($1^x = 1$) or less (if $b < 1$, then b^x gets closer and closer to 0 as x gets larger).

When a function is built from a sum of terms, the fastest growing term wins in the race to ∞. So in $3x^3 - x^2 + 7x + 5$ the x^3 term is the fastest growing term, so it "wins" and this polynomial goes to ∞ as $x \to \infty$ (x goes to ∞). Likewise, $-x^4 + 15x^2 + 9$ is dominated by $-x^4$, so this polynomial goes to $-\infty$ as $x \to \infty$.

Your turn: Let $f(x) = 5 - 6x + 2x^4 - \sqrt{x} + 99x^3$. Which term "wins", that is, goes to $\pm\infty$ fastest? Does $f(x)$ go to $+\infty$ or $-\infty$ as $x \to \infty$?

Your turn: Let $g(x) = 999x^{0.999} - x$. Which term "wins", that is, goes to $\pm\infty$ fastest? Does $g(x)$ go to $+\infty$ or $-\infty$ as $x \to \infty$?

You **can** divide by zero!

Ok. Not really. Division by zero causes all sorts of trouble. That is why mathematicians have decided to leave this operation undefined. Although we cannot divide by zero, we can see what happens if we divide by something *close* to zero.

Experiment: See what happens when you compute $\frac{1}{0.1}, \frac{1}{0.01}, \frac{1}{0.0001}, \frac{1}{0.000001}, \ldots$. What happens if you compute $\frac{1}{-0.1}, \frac{1}{-0.01}, \frac{1}{-0.0001}, \frac{1}{-0.000001}, \ldots$?

Let's refer to a very, very, very small positive number (if fact so small it's really zero) as 0^+ and a very, very, very small negative number (if fact so small it's really zero) as 0^-. Then the above experiment tells us we should expect $\frac{1}{0^+} = +\infty$ and $\frac{1}{0^-} = -\infty$.

Experiment: See what happens when you compute $\frac{1}{10}, \frac{1}{100}, \frac{1}{10000}, \frac{1}{1000000}, \ldots$. What happens if you compute $\frac{1}{-10}, \frac{1}{-100}, \frac{1}{-10000}, \frac{1}{-1000000}, \ldots$?

The above experiment tells us how we could define the notion of "*dividing* by infinity". In particular, $\frac{1}{\infty} = 0^+$ and $\frac{1}{-\infty} = 0^-$.

If we ignore plus and minus signs, we have discovered that...

$$\frac{1}{\text{small}} = \text{Big} \qquad \text{and} \qquad \frac{1}{\text{Big}} = \text{small}$$

Putting this together with what we learned on the last page, we can compute limits to infinity. Keep in mind, only the dominant term matters when heading to infinity.

Example: $\lim_{x\to\infty} \frac{1}{x^3-5x^2+1} = \lim_{x\to\infty} \frac{1}{x^3} = \frac{1}{+\infty} = 0$

Example: $\lim_{x\to\infty} -6x^4 + 99x^3 - 7x^2 - 4 = \lim_{x\to\infty} -6x^4 = -\infty$

Example: $\lim_{x\to\infty} \frac{3x^2-5x+2}{-6x^3+17x^2+3x-1} = \lim_{x\to\infty} \frac{3x^2}{-6x^3} = \lim_{x\to\infty} \frac{1}{-2x} = \frac{1}{-\infty} = 0$

Example: $\lim_{x\to\infty} \frac{10x^4-3x^2-112}{5x^4-13x^3+8x} = \lim_{x\to\infty} \frac{10x^4}{5x^4} = \lim_{x\to\infty} 2 = 2.$

So if the numerator (the top) has the largest term, we fly off to $\pm\infty$, and if the denominator (the bottom) has the largest term, we head to 0. If there is a tie, we head toward the ratio of the leading coefficients.

Your Turn:

$$\lim_{x\to\infty} \frac{6x^2 - x + 9}{-4x^3 + \sqrt{x} + 5} =$$

$$\lim_{x\to\infty} \frac{-2x + 6}{4x + 77} =$$

$$\lim_{x\to\infty} \frac{1 - 9x^3}{5 + 123\sqrt{x}} =$$

$$\lim_{x\to\infty} \frac{-8x + x^3 + 3}{2^x - 5x^2} =$$

$$\lim_{x\to\infty} \frac{6x^2 + x^4 - 9x^5 + x - 77}{3x^5 + 2} =$$

$$\lim_{x\to\infty} \frac{3^x + x^2 - 1}{2^x + x^4 - 5} =$$

Graphs can tell us a lot about what a function is *really* like. The behavior of a graph near its roots is quite important in many applications. Let's explore this topic in the context of polynomials. First, a few definitions.

A **root** of a function $f(x)$ is a number x such that $f(x) = 0$.

Roots are easy to spot on a graph. Graphically, a root is a place where the graph of $f(x)$ touches or crosses the x-axis.

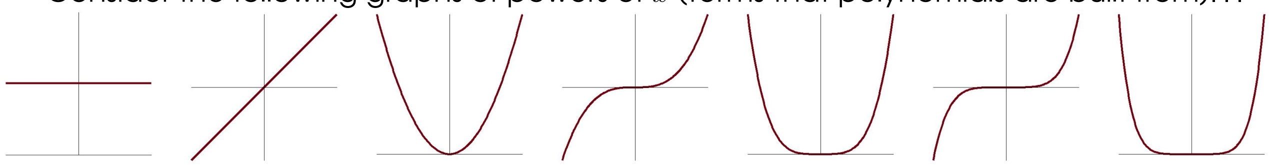
We will be focusing on graphs of polynomials. So let's define what a polynomial is.

A **polynomial** is a sum of terms like Cx^k where C is a real number and $k = 0, 1, 2, \ldots$.

Example: $f(x) = 5x^4 - 3x^2 + 7x + 10$ and $g(x) = -8x + 3$ are polynomials.

Non-Example: $h(x) = x^2 - 5\sqrt{x} + 3$ is not a polynomial.
(Because $\sqrt{x} = x^{1/2}$ is not an integral power of x.)

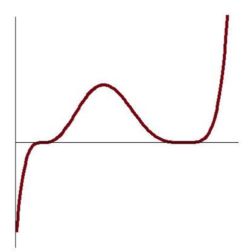
Consider the following graphs of powers of x (terms that polynomials are built from)...



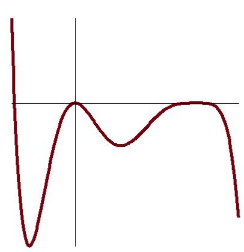
$y = x^0 = 1$ $y = x^1 = x$ $y = x^2$ $y = x^3$ $y = x^4$ $y = x^5$ $y = x^6$

Once we get past $y = 1$ and $y = x$, we can see a clear pattern of U-shaped graphs (for even powers) and S-shaped graphs (for odd powers). Other than $y = 1$ all of these graphs have a root at $x = 0$. Notice that if the root is repeated an odd number of times, the graph *crosses* the x-axis. If the root is repeated an even number of times, it *bounces* off of the x-axis. This same behavior holds for roots in general.

Example: $y = (x-1)^3(x-5)^4$ has 2 roots: $x = 1$ and $x = 5$. The root $x = 1$ is repeated 3 times, so the graph crosses the x-axis at $x = 1$. On the other hand, the root $x = 5$ is repeated 4 times, so the graph bounces off of the x-axis at $x = 5$. Notice also, that as $x \to \infty$ we have $y \to \infty$. Putting this information together, we can sketch a plot of our polynomial!



$y = (x-1)^3(x-5)^4$.

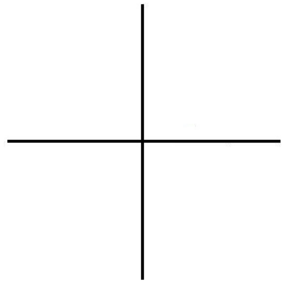


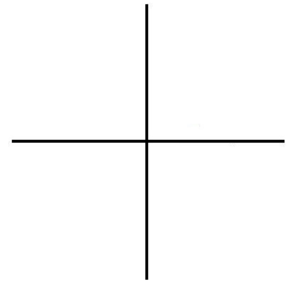
$y = -5x^2(x+1)(x-2)^4$

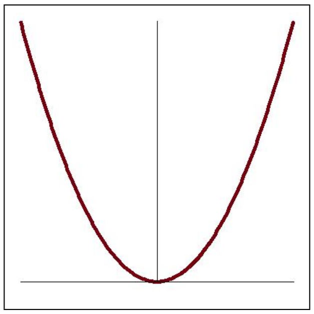
Example: $y = -5x^2(x+1)(x-2)^4$ has 3 roots: $x = -1$, $x = 0$, and $x = 2$. The root $x = -1$ appears only once, so the graph crosses the x-axis at $x = -1$. The root $x = 0$ is repeated twice and $x = 2$ is repeated 4 times, so the graph bounces off the x-axis at $x = 0$ and $x = 2$. Finally, the leading coefficient is -5 so as $x \to \infty$, $y \to -\infty$. Putting this information together, we can sketch a plot of our polynomial!

Your Turn:

Make a quick sketch of... $y = x^7$ and $y = x^8$.







Could the plot above be a sketch of $y = x^9$?

How about $y = x^{10}$?

Consider $f(x) = -2(x+1)^3x^4(x-1)(x-2)^5$.

Below each root write either "bounce" or "cross". Also, $\lim_{x\to\infty} f(x) =$

Root	$x = -1$	$x = 0$	$x = 1$	$x = 2$
Bounce or Cross?				

Sketch the graph of $y = f(x)$.

Sketch the graph of $y = (x+2)^2(x-1)^3$.

Sketch $y = -3(x-1)(x-3)^3$.

Sketch $y = -(x+2)^3x^2(x-2)^2(x-4)^3$.

Today we will build off of our previous work. Let's look at graphs of rational functions.

A **rational function** is a function of the form "polynomial divided by non-zero polynomial".

Example: $f(x) = \dfrac{5x^2 - 2x + 1}{3x - 6}$ and $g(x) = \dfrac{5}{x^3 - 1}$ are rational functions.

Also, all polynomials are rational functions (they are the ratio of that polynomial and the constant polynomial "1").

Now let's define the term "asymptote". To be precise, we are defining what *vertical* asymptotes are (there are other kinds of asymptotes).

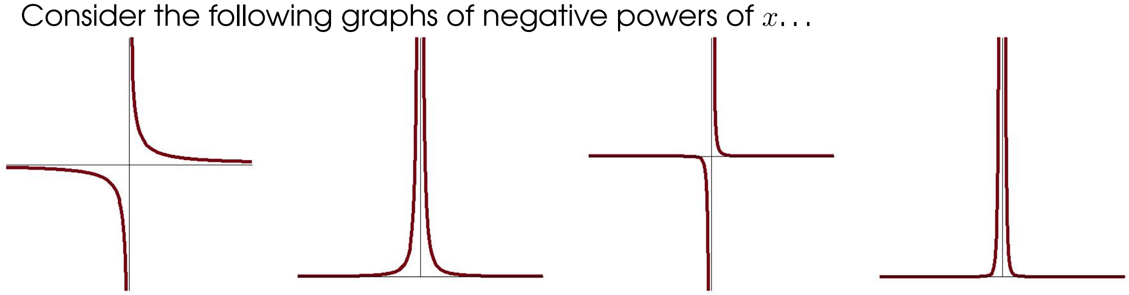
An **asymptote** of a function $f(x)$ is a line $x = a$ such that $f(x)$ approaches $x = a$ as $x \to a$.

Like roots, vertical asymptotes are easy to spot on a graph. Graphically, at a vertical asymptote the graph of $f(x)$ will fly off toward $\pm\infty$. Algebraically, we get an asymptote everywhere we have division by zero (assuming that our rational function is reduced, that is, no factors can be canceled off).

Example: $h(x) = \dfrac{-2(x-3)}{x^2(x-5)^3}$ is a rational function whose only root is $x = 3$. On the other hand, $h(x)$ has 2 vertical asymptotes, one at $x = 0$ and another at $x = 5$.

Non-Example: $k(x) = \dfrac{4(x-3)}{(x-3)}$ has no asymptotes. If we cancel off $(x-3)$, we get $k(x) = 4$.

Consider the following graphs of negative powers of x...

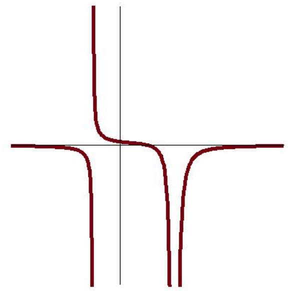


$y = \frac{1}{x}$ $\quad$ $y = \frac{1}{x^2}$ $\quad$ $y = \frac{1}{x^3}$ $\quad$ $y = \frac{1}{x^4}$

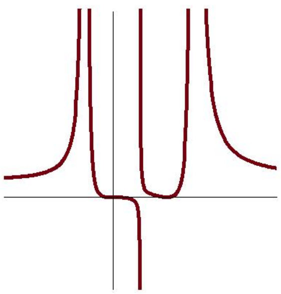
All of these rational functions have the asymptote $x = 0$. But notice that odd powers have branches going in different directions while even powers have branches going in the same direction. This can be explained by our infinite arithmetic from day #2. Recall that $\frac{1}{0^-} = -\infty$ and $\frac{1}{0^+} = +\infty$. Consider $1/x^2$. We have $1/(0^-)^2 = 1/0^+ = +\infty$ and $1/(0^+)^2 = 1/0^+ = +\infty$, so $1/x^2$ approaches $+\infty$ from both sides. On the other hand, $1/(0^-)^3 = 1/0^- = -\infty$ and $1/(0^+)^3 = 1/0^+ = +\infty$, so $1/x^3$ approaches $-\infty$ on the negative (left) side of zero and $+\infty$ on the positive (right) side of zero. We sum this up as follows: Asymptotes associated with even powers <u>bounce</u> off of infinity while asymptotes associated with odd powers <u>cross</u> over infinity.

One last thing to notice is that all of these graphs have a horizontal asymptote: $y = 0$ (the x-axis). This is because $\lim\limits_{x \to \pm\infty} \frac{1}{x^k} = 0$ where $k = 1, 2, \ldots$. In general, if $\lim\limits_{x \to \infty} f(x) = a$, then $y = f(x)$ has a **<u>horizontal asymptote</u>** $y = a$. A rational function can have at most one horizontal asymptote. If such an asymptote exists, the rational function approaches it as $x \to \infty$ and $x \to -\infty$. Armed with all of these tools, we're ready to plot rational functions.

Example: Let's sketch a plot of $y = \dfrac{-2(x-1)}{(x+1)(x-2)^2}$.



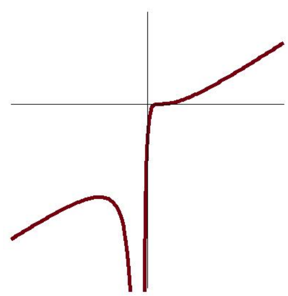
First, notice that $x = 1$ is a root (a zero for the top of the fraction). Since its multiplicity is 1, our graph will cross the x-axis at $x = 1$.
Next, $x = -1$ and $x = 2$ are asymptotes (zeros of the bottom of the fraction). Since $x = -1$ isn't repeated, our graph will "cross infinity" at $x = -1$. On the other hand $x = 2$ is repeated 2 times, so our graph will "bounce off of infinity" at $x = 2$.
Finally, notice that the degree of the top of our fraction is 1 and the degree of the bottom of our fraction is $1 + 2 = 3$. Since the degree of the bottom is bigger than the degree of the top, the bottom dominates and $y \to 0$ as $x \to \infty$. Thus $y = 0$ is a horizontal asymptote. All of this allows us to sketch our function.



Example: Let's sketch a plot of $y = \dfrac{2x^3(x-2)^2}{(x+1)^2(x-1)(x-3)^2}$.

First, notice that $x = 0$ and $x = 2$ are roots (zeros of the top of the fraction). Since $x = 0$ is repeated 3 times, we cross the x-axis at $x = 0$. On the other hand, $x = 2$ is only repeated 2 times, so we bounce off of the x-axis at $x = 2$.
Next, $x = -1$, $x = 1$, and $x = 3$ are asymptotes (zeros of the bottom of the fraction). Since $x = -1$ is repeated 2 times, we'll bounce off of infinity at $x = -1$. The same is true for $x = 3$. Notice that $x = 1$ has multiplicity 1, so we will cross infinity at $x = 1$.
Finally, notice that the degree of the top of our fraction is $3 + 2 = 5$ and the degree of the bottom of our fraction is $2 + 1 + 2 = 5$. Since these match, the limit of y as $x \to \infty$ is $2/1 = 2$ (the leading coefficient of the top divided by that of the bottom). This means that our graph has the horizontal asymptote $y = 2$. Now we can make a good sketch.

Example: Let's sketch a plot of $y = \dfrac{5(x-2)^3}{(x+1)^2}$.



The top of our fraction tells us we have a single root $x = 2$ which is repeated 3 times (so we cross the x-axis at $x = 2$). The bottom of our fraction has a single zero, so we get a single asymptote at $x = -1$. This is repeated twice, so we bounce off of infinity at $x = -1$. Finally, the degree of the top is 3 and the degree of the bottom is 2. Since the top dominates and we have a positive leading coefficient, $y \to \infty$ as $x \to \infty$. Now we can sketch our graph.

Your Turn: $y = \dfrac{3(x-1)^2}{x^2(x+1)(x-3)^4}$

Roots?		Asymptotes?	
Bounce or Cross?		Bounce or Cross?	

Sketch $y = \dfrac{-1}{(x-1)^3}$.

Sketch $y = \dfrac{3x^2}{(x+1)^2}$.

Sketch $y = \dfrac{-2(x+1)^2(x-2)}{x(x-1)^2}$.

Sketch $y = \dfrac{(x-2)^6}{(x+1)^2(x-1)^3}$.

Sketch $y = \dfrac{x(x-1)^3}{(x+1)^2(x+2)^3(x-2)}$.

Sketch $y = \dfrac{(x+1)(x-2)^2(x-3)^3}{x^2(x-1)}$.