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Free Leibniz Algebras

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Abstract

Leibniz algebras are a generalization of Lie algebras. While Lie algebras are well-known, Leibniz algebras are still in development. Our main goal is to examine the free Lie and free Leibniz algebras. In doing so, we also discuss free objects in general. The paper concludes by examining a fairly new result for free Leibniz algebras discovered by [MI].

1 Introduction

Lie algebras were introduced by S. Lie in the 1800s. He studied Lie algebras to gain insight into group theory. In particular, Lie algebras are related to a group, called the Lie group. Other mathematicians, such as E. Cartan and W. Killing, began to study Lie algebras, and their efforts helped bring about a complete classification of finite dimensional simple Lie algebras [CA]. Leibniz algebras, on the other hand, are a fairly new algebra. They were introduced by J. Loday in the 1990s [LO]. Leibniz algebras are a generalization of Lie algebras, and it was recently discovered that there are analogs of Lie's Theorem, Engel's Theorem, Cartan's criterion, and Levi's Theorem for Leibniz algebras [DE].

The goal of this paper is to discuss free Lie algebras and free Leibniz algebra. Essentially, we start from the ground up by examining free objects in general. We define free objects in terms of their universal mapping properties while refraining from the categorical definition of a free object as being left adjoint to the forgetful functor. Therefore, a working knowledge of undergraduate modern algebra and linear algebra should be all that is necessary to follow along in the paper. We conclude by discussing the basic properties of Lie and Leibniz algebras, and emphasize certain results pertaining to the universal enveloping algebra for free Lie algebras, and the basis for free Leibniz algebras.

2 A Survey of Free Objects

Let $\mathcal{C}$ be at category of algebraic objects and morphisms. We may think of $\mathcal{C}$ as containing groups with group homomorphisms or real vector spaces with linear maps. Definition 2.1 states the requirements for an algebraic object in $\mathcal{C}$ to be free.

Definition 2.1. *Let X be an arbitrary set. Given an object $F(X)$ and a function $i : X \to F(X)$, we say that $F(X)$ is **free** on X if given any object A and any function $\varphi : X \to A$, there exists a unique morphism $\widehat{\varphi} : F(X) \to A$ such that $\widehat{\varphi} \circ i = \varphi$.*

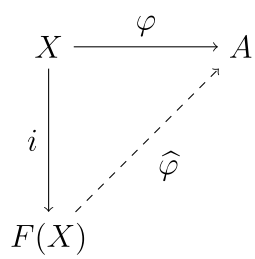

Figure 1: The commutative diagram for $F(X)$.

Figure 1 represents the theorem diagrammatically; the function $\widehat{\varphi}$ is represented using a dotted line to indicate the existence of $\widehat{\varphi}$ must be proved and is therefore not part of the initial conditions of the definition. Because of the universal quantifiers on φ and A in definition 2.1, we say that the free object $F(X)$ on X satisfies the **universal mapping property**, which we hereafter refer to as the UMP. We list several properties of free objects that are all consequences of the UMP.

Theorem 2.1. *Let X and Y be arbitrary sets and $\mathcal{C}$ be a category that contains free objects. If $F(X)$ is the free object on X, and $F(Y)$ is the free object on Y, then*

1. *$F(X)$ is unique up to isomorphism;*

2. *if $|X| = |Y|$, then $F(X) \cong F(Y)$;*

3. *every object in $\mathcal{C}$ is a homomorphic image of a free object in $\mathcal{C}$.*

Proof of Theorem 2.1.

1. Assume there are two free objects on X, say $F_1(X)$ and $F_2(X)$ with $i_1 : X \to F_1(X)$ and $i_2 : X \to F_2(X)$. The universal mapping property of $F_1(X)$ implies there is a unique morphism $\widehat{\varphi}_1$ from $F_1(X) \to F_1(X)$ such that $\widehat{\varphi}_1 \circ i_1 = i_1$. But, since the identity map $Id_{F_1(X)}$ is also a morphism from $F_1(X)$ to $F_1(X)$ such that $Id_{F_1(X)} \circ i_1 = i_1$, it follows from the uniqueness of $\widehat{\varphi}_1$ that

$$\widehat{\varphi}_1 \circ i_1 = Id_{F_1(X)} \circ i_1 = i_1. \tag{1}$$

That is, the identity map on $F_1(X)$ is the only morphism that can satisfy equation 1. Similarly, the identity map on $F_2(X)$ is the only morphism such that

$$Id_{F_2(X)} \circ i_2 = i_2. \tag{2}$$

Next, the universal mapping property of $F_1(X)$ implies there exists a unique morphism $\widehat{\varphi}_1 : F_1(X) \to F_2(X)$ such that

$$\widehat{\varphi}_1 \circ i_1 = i_2.$$

The universal mapping property of $F_2(X)$ implies there exists a unique morphism $\widehat{\varphi}_2 : F_2(X) \to F_1(X)$ such that

$$\widehat{\varphi}_2 \circ i_2 = i_1.$$

Consequently,

$$\widehat{\varphi}_1 \circ i_1 = \widehat{\varphi_1} \circ \widehat{\varphi}_2 \circ i_2 = i_2.$$

Equation 2 implies that $\widehat{\varphi_1} \circ \widehat{\varphi}_2 = Id_{F_2(X)}$. In a similar fashion, we have

$$\widehat{\varphi}_2 \circ i_2 = \widehat{\varphi_2} \circ \widehat{\varphi}_1 \circ i_1 = i_1.$$

Equation 1 implies that $\widehat{\varphi_2} \circ \widehat{\varphi}_1 = Id_{F_1(X)}$. Hence, $\widehat{\varphi_1}$ and $\widehat{\varphi_2}$ are inverses of each other, and $F_1(X)$ is isomorphic to $F_2(X)$.

2. Since free objects are unique up to isomorphism, we show that $F(X)$ also satisfies the universal mapping property of Y. There are bijections $f : X \to Y$ and $f^{-1} : Y \to X$. Let A be an object and $\varphi : Y \to A$ a map. Because $F(X)$ is free on X, there is a map $i : X \to F(X)$ and a unique morphism $\widehat{\varphi} : F(X) \to A$ such that

$$\widehat{\varphi} \circ i = \varphi \circ f.$$

This implies that

$$\widehat{\varphi} \circ i \circ f^{-1} = \varphi.$$

Hence, $F(X)$, when paired with the map $i \circ f^{-1} : Y \to F(X)$, is free on Y, so $F(X) \cong F(Y)$.

3. Let A be an object and $F(A)$ the free object on A (note that A can be thought of as an generating set.) Then there is a map $i : A \to F(A)$ and a unique morphism $\widehat{\varphi} : F(A) \to A$ such that

$$\widehat{\varphi} \circ i = Id_A,$$

where Id_A is the identity map on A. This implies that $\widehat{\varphi}$ is surjective as desired.

□

The existence clause of the UMP (that is, the existence of $\widehat{\varphi}$) implies there are no nontrivial relations in $F(X)$, while the uniqueness of $\widehat{\varphi}$ implies the elements of X generate $F(X)$. When we say there no nontrivial relations in $F(X)$, we mean that every relation that exists in $F(X)$ can be implied solely from the axioms that define the object. For example, if $F(X)$ is the free group on X, then for a collection of elements $g_i \in F(X)$, if we have the relation $g_{i_1} \cdot g_{i_2} \cdot ... \cdot g_{i_n} = g_{j_1} \cdot g_{j_2} \cdot ... \cdot g_{j_m}$, then the axioms that define a group are all that suffice to prove this relation exists. One may therefore think of free objects as having no imposed relations.

The elements of X generate $F(X)$ in the sense that every element in $F(X)$ can be expressed as a combination of elements in X. For example, if $F(X)$ is the free vector space on X, then for all $y \in F(X)$, $y \in \text{span}(x_1, x_2, ..., x_n)$, where each $x_i \in X$. We now proceed to explore examples of free objects for specific categories. The properties of $F(X)$ in theorem 2.1 apply to all free objects – assuming they exist in the first place. Indeed, some categories do not have free objects, such as fields, which we discuss in section 2.5.

2.1 Free Monoids

The free monoid is a monoid that satisfies the UMP of definition 2.1. Let X be a set, and define $M(X)$ to be the set of all associative words on X. For example, if $X = \{a, b, c\}$, then

$$a, \quad ab, \quad \text{and} \quad ccabac$$

are examples of associative words on X. Associative words on X are just finite sequences of elements of X. We call elements of $M(X)$ associative words to indicate that there is no need to parenthesize the words. Later on, when we study objects that can be non-associative, such as algebras, the need to distinguish between associative and non-associative words will be more clear. Equip $M(X)$ with the binary operation $* : M(X) \times M(X) \to M(X)$ given by $w_1 * w_2 = w_1 w_2$. The operation is just the concatenation of words, which is associative. Notice that $*$ is not commutative because we are treating elements like ab and ba as distinct words. $M(X)$ is a monoid, where the empty word, denoted ε, is the unit. Notice that elements of X are also words on X – we call them singleton words. We are therefore justified in defining the inclusion map

$$i : X \to M(X),$$

which is given by $i(x) = x$. We claim that $M(X)$ is the free monoid on the set X.

Theorem 2.2. *$M(X)$ is the free monoid on the set X.*

Proof. Let A be a monoid and $\varphi : X \to A$ be a map. We must show there exists a unique monoid homomorphism $\widehat{\varphi}$ from $M(X)$ to A such that $\widehat{\varphi} \circ i = \varphi$. For a nonempty word $w = x_1 x_2 ... x_n \in M(X)$, define $\widehat{\varphi} : M(X) \to A$ by

$$\widehat{\varphi}(w) = \widehat{\varphi}(x_1 x_2 ... x_n) = \varphi(x_1) \cdot \varphi(x_2) \cdot ... \cdot \varphi(x_n),$$

where $\cdot$ denotes the operation of A. If w is the empty word, then define $\widehat{\varphi}(\varepsilon) = e_A$, where e_A is the unit of A. Then $\widehat{\varphi}$ is a homomorphism such that $\widehat{\varphi} \circ i = \varphi$. It remains to show that $\widehat{\varphi}$ is unique. Suppose that $\bar{\varphi}$ is another homomorphism from $M(X)$ to A such that $\bar{\varphi} \circ i = \varphi$. Then for any nonempty word $w = x_1 x_2 ... x_n \in X$, we have

$$\begin{aligned}
\bar{\varphi}(w) &= \bar{\varphi}(x_1 * x_2 * ... * x_n). \\
&= \bar{\varphi}(x_1) \cdot \bar{\varphi}(x_2) \cdot ... \cdot \bar{\varphi}(x_n). \\
&= \bar{\varphi} \circ i(x_1) \cdot \bar{\varphi} \circ i(x_2) \cdot ... \cdot \bar{\varphi} \circ i(x_n). \\
&= \varphi(x_1) \cdot \varphi(x_2) \cdot ... \cdot \varphi(x_n). \\
&= \widehat{\varphi} \circ i(x_1) \cdot \widehat{\varphi} \circ i(x_2) \cdot ... \cdot \widehat{\varphi} \circ i(x_n). \\
&= \widehat{\varphi}(x_1) \cdot \widehat{\varphi}(x_2) \cdot ... \cdot \widehat{\varphi}(x_n). \\
&= \widehat{\varphi}(x_1 * x_2 * ... * x_n). \\
&= \widehat{\varphi}(w).
\end{aligned}$$

Also, $\widehat{\varphi}(\varepsilon) = \bar{\varphi}(\varepsilon)$. Thus, $\widehat{\varphi}$ is unique, and $M(X)$ is the free monoid on X. □

Example 2.1. Let $X = \{1\}$. Then the free monoid on X is just the set of all finite sequences of ones and the empty word: $M(X) = \{\varepsilon, 1, 11, 111, 1111, ...\}$. Let $\mathbb{Z}_{\geq 0}$ be the monoid of nonnegative integers under addition. Then $M(X) \cong \mathbb{Z}_{\geq 0}$.

Example 2.2. Let $X = \emptyset$ be the empty set. For any monoid A, the maps $j : X \to M(X)$ and $\varphi : X \to A$ are the canonical empty functions. Here, $M(X)$ is the set containing the empty word, and $\widehat{\varphi}$ is the trivial homomorphism that sends the empty word to the identity element in A. Note the requirement that $\widehat{\varphi} \circ j = \varphi$ is vacuously true. Also, the requirement that $\widehat{\varphi}$ is unique stems from the fact that our only element is the identity and a homomorphism must send the identity to the identity. Thus there is only one possible homomorphism from the trivial monoid.

In the next section, we explore free groups. The construction of the free monoid was fairly simply, and although the constructions of other free objects are similar in some ways, the reader may notice that as the complexity of the algebraic object increases, the tools needed to construct the free object become more advanced.

2.2 Free Groups

The construction of the free group on a set X is similar to that of the free monoid. For an arbitrary set $X = \{x_1, x_2, ...\}$, define $X^{-1} = \{x_1^{-1}, x_2^{-1}, ...\}$. That is, X^{-1} is the set obtained from X by inserting the superscript, $^{-1}$, on each of the elements of X. Let $G(X)$ be the set of all associative words on $X \cup X^{-1}$. This time, we denote the empty word by e_G. We allow for words in $G(X)$ to be concatenated, but concatenation alone is not enough to ensure that $G(X)$ is a group since we are treating elements like xx^{-1} and $x^{-1}x$ as distinct words. We deal with inverses by defining an equivalence relation.

Definition 2.2. *Let X be a set. Let $G(X)$ denote the set of all associative words obtained from $X \cup X^{-1}$. For $w, u \in G(X)$, if u can be obtained from w by a finite sequence of insertions or deletions of words of the form xx^{-1} or $x^{-1}x$, where $x, x^{-1} \in X \cup X^{-1}$, then we say w is **related** to u and write $w \sim u$.*

For example, if $X = \{a, b, c, d\}$, then

$$\begin{aligned} a &\sim a. \\ a &\sim aaa^{-1}. \\ abc^{-1} &\sim b^{-1}babdd^{-1}c^{-1}aa^{-1}. \\ e_G &\sim d^{-1}db^{-1}bcc^{-1}a^{-1}a. \\ ba &\not\sim cbc^{-1}a. \end{aligned}$$

Hopefully, the examples above are enough to convince the reader that $\sim$ is an equivalence relation. For $w \in G(X)$, let $\overline{w} = \{u \in G(X) \mid w \sim u\}$. That is, $\overline{w}$ is the equivalence class of the word w. Let $\overline{G(X)}$ denote the set of all equivalence classes of words in $G(X)$. Then $\overline{G(X)}$ is a group under the operation $\overline{w} \cdot \overline{u} = \overline{wu}$, where wu is the word w and u concatenated. Although we do not prove it, the group operation is well-defined – see [RO]. We need to establish some notation. For $a \in X^{-1}$, we have that $a = b^{-1}$ for some $b \in X$. So, we let $a^{-1} = (b^{-1})^{-1} = b$. That is, $(b^{-1})^{-1} = b$ for all $b \in X$. Suppose w is a word, say $w = x_1x_2...x_n$, where $x_i \in X \cup X^{-1}$, then we define $w^{-1} = x_n^{-1}...x_2^{-1}x_1^{-1}$. It follows that the inverse of $\overline{w}$ is $\overline{w^{-1}}$, and the identity element is $\overline{e_G}$. The check for associativity is rather tedious; for a complete proof that $\overline{G(X)}$ satisfies all of the group axioms, see [RO].

Theorem 2.3. *$\overline{G(X)}$ is the free group on X.*

Proof. Let H be a group and $\varphi : X \to H$ be a map. First, we need a map from X to $\overline{G(X)}$. To this end, we can define the inclusion map

$$i : X \to \overline{G(X)}$$

by letting $i(x) = \overline{x}$. Next, we need a unique group homomorphism $\widehat{\varphi}$ from $\overline{G(X)}$ to H such that $\widehat{\varphi} \circ i = \varphi$. We do this rather carefully by first noticing that every nonempty word of $G(X)$ can be expressed as

$$w_1^{\epsilon_1} w_2^{\epsilon_2} ... w_n^{\epsilon_n},$$

where each $w_i \in X$ and $\epsilon_i \in \{-1, 1\}$. For all $e_G \neq \overline{w} \in \overline{G(X)}$, let

$$\widehat{\varphi} : \overline{G(X)} \to H$$

be given by

$$\widehat{\varphi}(\overline{w}) = \widehat{\varphi}(\overline{w_1^{\epsilon_1} w_2^{\epsilon_2} ... w_n^{\epsilon_n}}) = \varphi(w_1)^{\epsilon_1} * \varphi(w_2)^{\epsilon_2} * \cdots * \varphi(w_n)^{\epsilon_n},$$

where $*$ is the operation for H. Let $\widehat{\varphi}(e_G) = e_H$, where e_H is the identity element of H. It may not be obvious that $\widehat{\varphi}$ is well-defined. So, first observe that for $x, x^{-1} \in X \cup X^{-1}$,

$$\widehat{\varphi}(\overline{xx^{-1}}) = \varphi(x) * \varphi(x)^{-1} = e_H = \varphi(x)^{-1} * \varphi(x) = \widehat{\varphi}(\overline{x^{-1}x}).$$

Hence, $\widehat{\varphi}(\overline{xx^{-1}}) = \widehat{\varphi}(\overline{x^{-1}x}) = e_H$. With that in mind, if $\overline{w} = \overline{y} \in \overline{G(X)}$, then w and y must differ by the insertion or deletion of elements of the form xx^{-1} or $x^{-1}x$. Since the image of the equivalence classes of such elements under $\widehat{\varphi}$ are the identity element of H, it follows that $\widehat{\varphi}(\overline{w}) = \widehat{\varphi}(\overline{y})$. Hence, $\widehat{\varphi}$ is well-defined. It should be clear that $\widehat{\varphi}$ is a homomorphism such that $\widehat{\varphi} \circ i = \varphi$. It remains to show that $\widehat{\varphi}$ is unique, so let $\widehat{\varphi}'$ be another homomorphism such that $\widehat{\varphi}' \circ i = \varphi$. Then $\widehat{\varphi} \circ i = \widehat{\varphi}' \circ i$, and for $e_G \neq \overline{w} \in \overline{G(X)}$, we have

$$\begin{aligned}
\widehat{\varphi}(\overline{w}) &= \widehat{\varphi}(\overline{w_1^{\epsilon} w_2^{\epsilon} ... w_n^{\epsilon}}). \\
&= \widehat{\varphi}(w_1)^{\epsilon_1} * \widehat{\varphi}(w_2)^{\epsilon_2} * \cdots * \widehat{\varphi}(w_n)^{\epsilon_n}. \\
&= (\widehat{\varphi} \circ i(w_1))^{\epsilon_1} * (\widehat{\varphi} \circ i(w_2))^{\epsilon_2} * \cdots * (\widehat{\varphi} \circ i(w_n))^{\epsilon_n}. \\
&= (\widehat{\varphi}' \circ i(w_1))^{\epsilon_1} * (\widehat{\varphi}' \circ i(w_2))^{\epsilon_2} * \cdots * (\widehat{\varphi}' \circ i(w_n))^{\epsilon_n}. \\
&= \widehat{\varphi}'(w_1)^{\epsilon_1} * \widehat{\varphi}'(w_2)^{\epsilon_2} * \cdots * \widehat{\varphi}'(w_n)^{\epsilon_n}. \\
&= \widehat{\varphi}'(w_1^{\epsilon_1}) * \widehat{\varphi}'(w_2^{\epsilon_2}) * \cdots * \widehat{\varphi}'(w_n^{\epsilon_n}). \\
&= \widehat{\varphi}'(\overline{w_1^{\epsilon_1} w_2^{\epsilon_2} ... w_n^{\epsilon_n}}). \\
&= \widehat{\varphi}'(\overline{w}).
\end{aligned}$$

Also, $\widehat{\varphi}(e_G) = e_H = \widehat{\varphi}'(e_G)$. Hence, $\widehat{\varphi}$ is unique, and $\overline{G(X)}$ is the free group on X. □

Example 2.3. The integers under addition is a free group. To see why, let $X = \{1\}$. For notational convenience, let $a = 1$ and $b = 1^{-1}$. Then

$$\overline{G(X)} = \left\{\overline{e_G}, \overline{a}, \overline{b}, \overline{aa}, \overline{bb}, ...\right\}$$

is the free group on X. It is isomorphic to the group of integers of under addition.

Example 2.4. We show that the free group generated by more than one element is non-abelian. Let X be an arbitrary set with $|X| \geq 2$. We may write $X = \{a, b\} \cup Y$, where $Y \cap \{a, b\} = \emptyset$. Consider the symmetric group on three elements: $S_3 = \{(1), (12), (13), (23), (123), (132)\}$. We can define a map $\varphi : X \to S_3$ by letting

$$\varphi(x) = \begin{cases} (12) & x = a \\ (13) & x = b \\ (1) & x \in Y \end{cases}$$

We have a unique homomorphism $\widehat{\varphi} : \overline{G(X)} \to S_3$, but notice that

$$\widehat{\varphi}(\overline{ab}) = \widehat{\varphi}(\overline{a})\widehat{\varphi}(\overline{b}) = \varphi(a)\varphi(b) = (12)(13) = (132),$$

and

$$\widehat{\varphi}(\overline{ba}) = \widehat{\varphi}(\overline{b})\widehat{\varphi}(\overline{a}) = \varphi(b)\varphi(a) = (13)(12) = (123).$$

Since $\widehat{\varphi}(\overline{ab}) \neq \widehat{\varphi}(\overline{ab})$, we have $\overline{a}\overline{b} \neq \overline{b}\overline{a}$, which means $\overline{G(X)}$ is not abelian.

As with free monoids, the free group on the empty set is the trivial group. Free groups are interesting in their own right, and have useful applications to combinatorial group theory. For this paper, the purpose of discussing free monoids and free groups is to merely give concrete examples of how to construct free objects. Our goal is discuss free Lie algebras and free Leibniz algebras, and sections 2.3 and 2.4 provide the tools that we need to study those objects.

2.3 Vector Spaces Are Free

In this section, our main goal is to introduce the tensor product of vector spaces. To do so, we must discuss vector spaces. We work over arbitrary fields, and the results that we discuss are valid for both finite and infinite dimensional vector spaces.

Let X be a set and $\mathbb{F}$ a field. Let $V(X)$ be the set of functions from X into $\mathbb{F}$ such that only finitely many elements of x are mapped to zero. That is, $V(X) = \{f : X \to \mathbb{F} \mid f(x) \neq 0 \text{ for at most finitely many } x \in X\}$. For $f, g \in V(X)$, let $f+g$ be given by $(f+g)(x) = f(x)+g(x)$. Notice that $f+g$ is indeed in $V(X)$. For $c \in \mathbb{F}$, let cf be given by $(cf)(x) = cf(x)$. Again, it should be clear that $cf \in V(X)$. With these operations, $V(X)$ is a vector space. We omit the tedious proof that $V(X)$ satisfies all of the axioms for a vector space. We proceed by finding a basis for $V(X)$.

First, we define a characteristic function. For $x \in X$, let $\delta_x : X \to \mathbb{F}$ be given by $\delta_x(x) = 1$ and $\delta_x(y) = 0$ for all $y \neq x$. There is only one element whose image under δ_x is non-zero; therefore, $\delta_x \in V(X)$. We show that $\beta = \{\delta_x \mid x \in X\}$ is a basis for $V(X)$.

Let $f \in V(X)$. Then there are only a finite number of elements in X whose image under f is non-zero. Let $x_1, ..., x_n$ be those elements. Note that $f(x_1), ..., f(x_n) \in \mathbb{F}$. Define $g(x) = \sum_{i=1}^{n} f(x_i)\delta_{x_i}(x)$. Then we have

$$g(x_j) = \sum_{i=1}^{n} f(x_i)\delta_{x_i}(x_j) = 0 + ... + f(x_j) + ... + 0 = f(x_j).$$

This shows that f and g agree on $x_1, ..., x_n$. For $y \neq x_i$, we have

$$g(y) = \sum_{i=1}^{n} f(x_i)\delta_{x_i}(y) = 0 = f(y).$$

Hence, $f = g$, which proves that β spans $V(X)$.

As for linear independence, assume that $\sum_{i=1}^{n} c_i\delta_{x_i} = 0$, where $c_i \in \mathbb{F}$ and $x_i \in X$. Then for $j \leq n$, we have

$$0 = \sum_{i=1}^{n} c_i\delta_{x_i}(x_j) = c_j.$$

Hence, β is linearly independent, and so β is a basis for $V(X)$. The proof that $V(X)$ is free on X follows quickly.

Theorem 2.4. *$V(X)$ is the free vector space on X.*

Proof. Let $i : X \to V(X)$ be defined by $i(x) = \delta_x$, where δ_x is the characteristic function from earlier. Let W be a vector space and $\varphi : X \to W$ a map. For any $f = \sum_{i=1}^{n} c_i\delta_{x_i} \in V(X)$, let $\widehat{\varphi} : V(X) \to W$ be given by

$$\widehat{\varphi}(f) = \sum_{i=1}^{n} c_i\varphi(x_i).$$

It should be apparent that $\widehat{\varphi}$ is a linear map such that $\widehat{\varphi} \circ i = \varphi$. Hence, $V(X)$ is free on X.

□

In the proof of theorem 2.4, we see that every vector space is free on its basis. Since every vector space has a basis, all vector spaces are therefore free. Indeed, if V is a vector space and $\beta = \{b_1, b_2, ...\}$ a basis for V, then for any map φ from β into a vector space W, there exists a unique linear map from V to W given by

$$\widehat{\varphi}(v) = \sum_{i=1}^{n} c_i \varphi(b_i).$$

We say that any map defined on a basis may be extended by linearity to a unique linear map defined on the vector space spanned by the basis. So, from this point on, we bypass saying that $V(X)$ is the free vector space on X, and instead simply say that $V(X)$ is the vector space with basis X.

2.3.1 The Tensor Product

In this section, we define the tensor product of vector spaces and sketch a formal construction. Before doing so, we give the definition of a multilinear map.

Definition 2.3. *Let $V_1, V_2, ..., V_n$, and W be vector spaces over a field $\mathbb{F}$. We say that $\gamma : V_1 \times V_2 \times \cdots \times V_n \to W$ is a* ***multilinear*** *map if for all $v_1 \in V_1, ..., v_i, v_i' \in V_i, ..., v_n \in V_n$ and $c \in \mathbb{F}$,*

1. $\gamma(v_1, ..., v_i + v_i', ..., v_n) = \gamma(v_1, ..., v_i, ..., v_n) + \gamma(v_1, ..., v_i', ..., v_n)$;
2. $\gamma(v_1, ..., cv_i, ..., v_n) = c\gamma(v_1, ..., v_i, ..., v_n)$.

We see that a multilinear map is map defined on a cartesian product of vector spaces that is linear in each of is arguments when all other arguments are fixed. A multilinear map defined a two-fold cartesian product is called **bilinear**. Similar to linear maps, if $\beta_1, ..., \beta_n$ are bases for $V_1, ..., V_n$ respectively, then for any vector space W and any map $f : \beta_1 \times \cdots \times \beta_n \to W$, there exists a unique multilinear map $\gamma : V_1 \times \cdots \times V_n \to W$ such that γ and f agree on $\beta_1 \times \cdots \times \beta_n$.

Definition 2.4. *Let $V_1, ..., V_n$ be vector spaces over a field $\mathbb{F}$. Let $\mathcal{V}$ be a vector space over $\mathbb{F}$ and $T : V_1 \times \cdots \times V_n \to \mathcal{V}$ a multilinear map. We say that the vector space $\mathcal{V}$ equipped with the multilinear map T is a* ***tensor product*** *of $V_1, ..., V_n$ if, given a vector space W over $\mathbb{F}$ and a multilinear map $\gamma : V_1 \times \cdots \times V_n \to W$, there exists a unique linear map $\widehat{\gamma} : \mathcal{V} \to W$ such that $\widehat{\gamma} \circ T = \gamma$.*

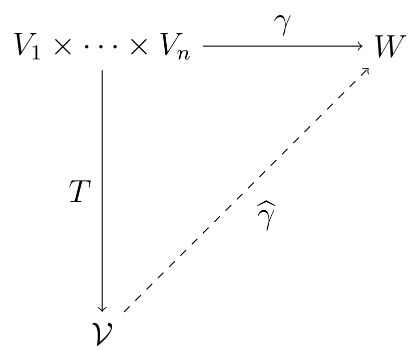

Figure 2: The commutative diagram for $\mathcal{V}$.

Note that a tensor product consists of a vector space equipped with a particular multilinear map. We may therefore refer to a tensor product as the pair $(\mathcal{V}, T)$. To prove the existence of the tensor product, one begins with the vector space $V(X)$ with basis $X = V_1 \times \cdots \times V_n$. Let S be the subspace of $V(X)$ spanned by elements of the form

$$(v_1, ..., v_i + v_i', ..., v_n) - (v_1, ..., v_i, ..., v_n) - (v_1, ..., v_i', ..., v_n)$$

and

$$(v_1, ..., cv_i, ..., v_n) - c(v_1, ..., v_i, ..., v_n),$$

where $v_i, v_i' \in V_i$ and $c_i \in \mathbb{F}$. Then the quotient space, $V(X)/S$, when equipped with the multilinear map $T : V_1 \times \cdots \times V_n \to V(X)/S$ given by $T(v_1, ..., v_n) = (v_1, ..., v_n) + S$, is a vector space that satisfies the universal mapping property of definition 2.4 and is therefore the tensor product of $V_1, ..., V_n$. For a complete proof that $V(X)/S$ satisfies the universal mapping property, see [CO]. It is customary to use different notation for tensor products. In particular, we let $V(X)/S = V_1 \otimes \cdots \otimes V_n$ denote the tensor product of $V_1, ..., V_n$. Elements of tensor products are cosets, but again for notational purposes, we set $T(v_1, ..., v_n) = (v_1, ..., v_n) + S = v_1 \otimes \cdots \otimes v_n$. We call the elements of a tensor product tensors. There are two properties of tensors that are inherit from the multilinearity of T. For each $v_i, v_i' \in V_i$ and $c \in \mathbb{F}$, we have the following.

1. $v_1 \otimes \cdots \otimes (v_i + v_i') \otimes \cdots \otimes v_n = v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n + v_1 \otimes \cdots \otimes v_i' \otimes \cdots \otimes v_n$.
2. $v_1 \otimes \cdots \otimes cv_i \otimes \cdots \otimes v_n = c(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n)$.

Moreover, one may show that tensor products are unique up to isomorphism.

Lemma 2.5. *Suppose that $(\mathcal{V}, T)$ and $(\mathcal{V}', T')$ are tensor products of the vector spaces $V_1, ..., V_n$ over $\mathbb{F}$. Then there exists an isomorphism $\phi : \mathcal{V} \to \mathcal{V}'$ such that $\phi \circ T = T'$.*

Proof. The universal mapping property of $\mathcal{V}$ implies there exists a unique linear map $\phi : \mathcal{V} \to \mathcal{V}'$ such that $\phi \circ T = T'$. Similarly, there is a unique linear map $\phi' : \mathcal{V}' \to \mathcal{V}$ such that $\phi' \circ T' = T$. This implies that $\phi' \circ \phi \circ T = T$, yet the uniqueness clause of the universal mapping property of $\mathcal{V}$ implies that $\phi' \circ \phi = Id_{\mathcal{V}}$. Similarly, $\phi \circ \phi' = Id_{\mathcal{V}'}$, which shows that $\mathcal{V}$ and $\mathcal{V}'$ are isomorphic. □

We use the lemma to show how to find a basis for the tensor product.

Theorem 2.6. *Let $V_1, ..., V_n$ be vector spaces over a field $\mathbb{F}$ with bases $\beta_1, ..., \beta_n$ respectively. Then the set $B = \{b_1 \otimes \cdots \otimes b_n \mid b_i \in \beta_i\}$ is a basis for $V_1 \otimes \cdots \otimes V_n$.*

Proof. Set $X = V_1 \times \cdots \times V_n$ and $X' = \beta_1 \times \cdots \times \beta_n$. Referencing our previous notation, we have $V_1 \otimes \cdots \otimes V_n = V(X)/S$, where $V(X)$ is the vector space with basis X, and S is the subspace of $V(X)$ defined earlier. We also have the corresponding multilinear map $T : X \to V(X)/S$ defined earlier. Consider the vector space $V(X')$ with X' as basis. Then we have the inclusion maps $i_1 : X' \to X$ and $i_2 : X' \to V(X')$. The map i_2 can be extended to a unique multilinear map $T' : X \to V(X')$ such that $T' \circ i_1 = i_2$. We claim that $V(X')$ equipped with T' is also a tensor product of $V_1, ..., V_n$.

To that end, let γ be a multilinear map from X into some vector space W. Let γ' be the restriction of γ to X'. Then because X' is a basis for $V(X')$, γ' can be extended to a unique linear map $\widehat{\gamma'}$ from $V(X')$ to W such that $\widehat{\gamma'} \circ i_2 = \gamma'$. We will be done if we can show that $\widehat{\gamma'} \circ T' = \gamma$. Note that having $\widehat{\gamma'} \circ i_2 = \gamma'$ implies that $\widehat{\gamma'}$ and γ agree on X'. Since $\widehat{\gamma'}$ and γ are multilinear, they must also agree on X. Recall that $T' \circ i_1 = i_2$. This implies that $\widehat{\gamma'} \circ i_2 = \widehat{\gamma'} \circ T' \circ i_1 = \gamma'$, but since $\widehat{\gamma'}$ and γ agree on X, we have that $\widehat{\gamma'} \circ T' = \gamma$. Therefore, $V(X')$ equipped with T' is a tensor product for $V_1, ..., V_n$.

Ultimately, we know from lemma 2.5 that there is an isomorphism $\phi : V(X') \to V(X)/S$ such that $\phi \circ T' = T$. Since X' is a basis for $V(X')$, and

ϕ is an isomorphism, $\phi(X')$ must be a basis for $V(X)/S$. It follows that

$$\begin{aligned} b_1 \otimes \cdots \otimes b_n &= T(b_1, ..., b_n). \\ &= \phi \circ T'(b_1, ..., b_n). \\ &= \phi \circ T' \circ i_1(b_1, ..., b_n). \\ &= \phi \circ i_2(b_1, ..., b_n). \\ &= \phi(b_1, ..., b_n). \end{aligned}$$

Therefore, $\phi(X') = \{b_1 \otimes \cdots \otimes b_n \mid b_i \in \beta_i\}$ is a basis for $V_1 \otimes \cdots \otimes V_n$.

□

Example 2.5. Take $V_1 = \mathbb{R}$ and $V_2 = \mathbb{R}^2$. Then a basis for $V_1 \otimes V_2$ is given by $\{1 \otimes (0,1), 1 \otimes (1,0)\}$.

Example 2.6. Take $V_1 = \mathbb{R}^2$, and $V_2 = \mathbb{P}_{\leq 1}[x]$, where $\mathbb{P}_{\leq 1}[x]$ is the set of all polynomials with real coefficients of degree less than or equal to 1 and indeterminate x. Then $\{(0,1) \otimes 1, (0,1) \otimes x, (1,0) \otimes 1, (1,0) \otimes x\}$ is a basis for $V_1 \otimes V_2$.

If $V_1, ..., V_m$ are finite dimensional vector spaces with dimensions $n_1, ..., n_m$ respectively, then $V_1 \otimes \cdots \otimes V_n$ has dimension $n_1 \cdots n_m$. We also have the following theorems whose proofs can be found in [CO].

Theorem 2.7. *Let $V_1, ..., V_n$ and $W_1, ..., W_m$ be vector spaces over a field $\mathbb{F}$. Then $(V_1 \otimes \cdots \otimes V_n) \otimes (W_1 \otimes \cdots \otimes W_m) \cong V_1 \otimes \cdots \otimes V_n \otimes W_1 \otimes \cdots \otimes W_m$.*

Theorem 2.8. *Let $V_1, ..., V_n$ be vector spaces over a field $\mathbb{F}$ and π a permuation of $\{1, 2, ..., n\}$. Then $V_1 \otimes \cdots \otimes V_n \cong V_{\pi(1)} \otimes \cdots \otimes V_{\pi(n)}$ via a linear map that takes $v_1 \otimes \cdots \otimes v_m$ to $v_{\pi(1)} \otimes \cdots \otimes v_{\pi(n)}$.*

2.4 Free Algebras

After discussing algebras and some of their basic properties, we construct the free associative algebra. We rely on the free associative algebra in the construction of the free Lie algebra in section 3.3. Then, we construct the free non-associative algebra, which we use to construct the free Leibniz algebra in section 4.2.

2.4.1 Algebras

Vector spaces have two operations – vector addition and scalar multiplication. When equipped with a bilinear multiplication operator, a vector space becomes an algebra, and a plethora of new avenues to explore opens up as a consequence.

Definition 2.5. *An **algebra** A is a vector space over a field $\mathbb{F}$ equipped with a bilinear map, called multiplication, from $A \times A$ to A. That is, there exists a map $(,) : A \times A \to A$ such that*

1. $(x + y, z) = (x, z) + (y, z)$*;*
2. $(x, y + z) = (x, y) + (x, z)$*;*
3. $(cx, y) = (x, cy) = c(x, y)$ *for all* $x, y, z \in A$ *and* $c \in \mathbb{F}$*.*

The three conditions above follow directly from definition 2.3. The properties of a bilinear map agree with our natural inclination for how multiplication should work. For example, we have a property analogous to the classic FOIL method. For $x, y, w, z \in A$, we see that

$$\begin{aligned}(x + y, w + z) &= (x + y, w) + (x + y, z). \\ &= (x, w) + (y, w) + (x, z) + (y, z).\end{aligned}$$

We say that A is an associative algebra if $((x, y), z) = (x, (y, z))$ for all $x, y, z \in A$. For associative algebras, it is often convenient to denote the multiplication by juxtaposition. If $(x, y) = (y, x)$ for all $x, y \in A$, then A is a commutative algebra. If there is a vector $1 \in A$ such that $(1, x) = (x, 1) = x$ for all $x \in A$, then we say A is a unital algebra with unit 1.

Example 2.7. A field itself is an associative, commutative, and unital algebra.

Example 2.8. The vector space of all $n \times n$ matrices over a field with the usual operations of scalar multiplication, matrix addition, and matrix multiplication is an associative, unital algebra.

Example 2.9. When equipped with the cross-product, the three-dimensional vector space $A = \mathbb{R}^3$ becomes an algebra. Let $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ denote the standard unit vectors. Recall that the cross-product may be defined on the standard unit vectors by $-(\mathbf{j},\mathbf{i}) = (\mathbf{i},\mathbf{j}) = \mathbf{k}, -(\mathbf{k},\mathbf{j}) = (\mathbf{j},\mathbf{k}) = \mathbf{i}$, $-(\mathbf{i},\mathbf{k}) = (\mathbf{k},\mathbf{i}) = \mathbf{j}$, and $(\mathbf{i},\mathbf{i}) = (\mathbf{j},\mathbf{j}) = (\mathbf{k},\mathbf{k}) = 0$. This algebra is not associative, commutative, nor is it unital. It should be mentioned that it suffices to define an algebra's multiplication on the basis vectors. This is because bilinear maps (multilinear maps) are uniquely determined by their action on pairs (tuples) of basis vectors. Consequently, since $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ is a basis for A, one may verify that A is an algebra by checking that the vector cross-product rule is bilinear for the basis vectors. One can then extend the algebra's multiplication by bilinearity so that it is applied to all vectors in A. So, in general, knowing how an algebra's multiplication behaves on the basis determines how the multiplication behaves on the entire algebra.

Algebra homomorphisms behave as expected.

Definition 2.6. *Let A_1 and A_2 be algebras over a field $\mathbb{F}$. Then a linear map $\varphi : A_1 \to A_2$ is an algebra homomorphism if for all $x, y \in A_1$,*

$$\varphi(x \cdot y) = \varphi(x) * \varphi(y),$$

where $\cdot$ is the multiplication of A_1 and $*$ is the multiplication of A_2.

An associative algebra homomorphism is simply a homomorphism between associative algebras. Next, we discuss free associative algebras over a field.

2.4.2 The Tensor Algebra

For a vector space V over a field $\mathbb{F}$, let $T(V)^0 = \mathbb{F}, T(V)^1 = V, T(V)^2 = V \otimes V$, and in general

$$T(V)^n = V \otimes \cdots \otimes V \quad (n \text{ factors })$$

We say that $T(V)^n$ is the n^{th} tensor power of V. Consider the vector space of the direct sum of tensor powers of V given by

$$T(V) = \bigoplus_{n=0}^{\infty} T(V)^n = T(V)^0 \oplus T(V)^1 \oplus T(V)^2 \oplus \cdots$$

Elements of $T(V)$ are finite sums:

$$\mathbf{v}_0 + \mathbf{v}_1 + \cdots + \mathbf{v}_n,$$

where each $\mathbf{v}_k \in T(V)^k$ take on the form $v_1 \otimes \cdots \otimes v_k$ for $v_1, ..., v_k \in V$. Suppose $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 + \cdots + \mathbf{v}_n$ and $\mathbf{w} = \mathbf{w}_0 + \mathbf{w}_1 + \cdots + \mathbf{w}_m$ are elements of $T(V)$. Without loss of generality, suppose that $n \leq m$. For the sum $\mathbf{v} + \mathbf{w}$, we define addition component-wise. In the case where n is strictly less than m, we can insert a finite number of $\mathbf{0}'$s, where each $\mathbf{0} \in T(V)^k$ for some $k > n$, so that $\mathbf{v}$ and $\mathbf{w}$ have the same number of components. We define scalar multiplication component-wise as well.

If β is a basis for V, then a basis for $T(V)$ is given by

$$\mathbf{B} = \{b_1 \otimes \cdots \otimes b_n \mid n \geq 0, b_i \in \beta\}.$$

For $n = 0$, we denote the empty tensor product as 1. In particular, $b_0 = 1$ is the multiplicative identity of $T(V)^0 = \mathbb{F}$. Note that elements of $\mathbf{B}$ are members of $T(V)^j$ for some j. With that in mind, if we define a bilinear map from $T(V)^n \times T(V)^m$ to $T(V)^{n+m}$ satisfying

$$(v_1 \otimes \cdots \otimes v_n) \cdot (w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m, \qquad (3)$$

where $v_i, w_j \in V$, then we can extend this map by linearity to a multiplication map from

$$T(V) \times T(V) \to T(V).$$

With this multiplication map, $T(V)$ becomes an associative, unital algebra called the **tensor algebra** of V. The fact that $T(V)$ is an associative algebra follows from theorem 2.7. The unit of $T(V)$ is the element $b_0 = 1 \in T(V)^0$. Note the fact that (3) is bilinear follows directly from the properties of tensors given in section 2.3.1. The tensor algebra is the free associative algebra on any basis for V. We use the tensor algebra to construct the free associative algebra on any set X.

Theorem 2.9. *The free associative algebra over a field on a set X is the tensor algebra of the vector space with basis X.*

Proof. Let V be the vector space with basis X and $T(V)$ be the tensor algebra of V. We have the inclusion map $i : X \to T(V)$. Let A be an associative algebra, and $\varphi : X \to A$ a map. Since X is basis for V, the map φ can be extended to a unique linear map $\varphi' : V \to A$ such that $\varphi' \circ i = \varphi$. Since the basis vectors of V generate $T(V)$, the map φ' can be extended to a unique associative algebra homomorphism $\widehat{\varphi} : T(V) \to A$ such that $\widehat{\varphi} \circ i = \varphi$. $\square$

2.4.3 Free Non-associative Algebras

Let X be a set. Similar to how we handle free monoids and free groups, we consider the set $F(X)$ of all non-associative words on X. Non-associative words on X are finite sequences of elements of X that are parenthesized. We define the length of a word on X to be the number of elements of X that are used to construct the word. For instance, if $X = \{x, y\}$, then (x) is a non-associative word of length 1, (x, y) is a non-associative word of length 2, and $((x,(x,y)),y)$ is a non-associative word of length 4. We define the empty word ε to have length 0. Let $\mathcal{A}$ be the vector space with basis $F(X)$. That is, $\mathcal{A}$ is the vector space whose basis is the set of all non-associative words on X. We define a bilinear multiplication by concatenation. For example, if $w_1 = (x_1, x_2)$ and $w_2 = (y_1, y_2)$ are non-associative words on X, where $x_1, x_2, y_1, y_2 \in X$, then $w_1 \cdot w_2 = (w_1, w_2) = ((x_1, x_2), (y_1, y_2))$. With these operations, $\mathcal{A}$ becomes a non-associative unital algebra. The unit of $\mathcal{A}$ is the empty word ε. We show that $\mathcal{A}$ satisfies the universal property of free objects.

Theorem 2.10. *$\mathcal{A}$ is the free non-associative algebra on X.*

Proof. We have the inclusion map $i : X \to \mathcal{A}$. Let B be a unital, non-associative algebra with multiplication denoted by $*$ and $\varphi : X \to B$ a map. We need to construct a unique algebra homomorphism $\widehat{\varphi} : \mathcal{A} \to B$ such that $\widehat{\varphi} \circ i = \varphi$. Since the set of all non-associative words on X forms a basis for $\mathcal{A}$, defining how $\widehat{\varphi}$ acts on non-associative words uniquely determines how $\widehat{\varphi}$ behaves on $\mathcal{A}$. We define $\widehat{\varphi}$ to send the empty word to the unit of B. We proceed by induction on the length of the word. For $x \in X$, consider the non-associative word $w = (x)$, which is of length 1. We define $\widehat{\varphi}(w) = \varphi(x)$. Note that for any non-associative word w on X of length k, we can express w as the product of non-associative words w_1 and w_2 whose lengths are less than or equal to k. Continuing in that vein, we can uniquely express any non-empty, non-associative word w on X as a product of elements of X (singleton words of length 1). With that in mind, assume that $\widehat{\varphi}$ is defined for non-associative words of length less than n, where $n > 1$. Then for a non-associative word w' of length n, there exists non-associative words w'_1 and w'_2 of length less than n such that $w' = w'_1 \cdot w'_2$. The inductive hypothesis implies that $\widehat{\varphi}(w'_1) * \widehat{\varphi}(w'_2) = \widehat{\varphi}(w'_1 \cdot w'_2) = \widehat{\varphi}(w')$ is defined. Therefore, by induction, $\widehat{\varphi}$ is defined for all non-associative words on X. Note that the base case of the induction proof implies that $\widehat{\varphi} \circ i = \varphi$, the inductive hypothesis implies that $\widehat{\varphi}$ is a homomorphism, and the unique representation of any non-empty, non-associative word w on X as a product of elements of X ensures that $\widehat{\varphi}$ is well-defined. □

2.5 Free Rings and Free Fields

In this supplementary section, we discuss free rings, and we show why free fields do not exist. Our analysis of free Lie and free Leibniz algebras do not depend on a working knowledge of free rings or free fields, so the reader may skip this section without jeopardizing his or her understanding of the subsequent sections.

2.5.1 Free Rings

In this section we work in the category of unital rings (i.e., rings with multiplicative identity). Recall that in this category ring homomorphisms must send the multiplicative identity of the domain to the multiplicative identity of the codomain.

Let X be a set. Let X^* denote the set of all associative words on X. Recall that elements of X^* are finite sequences of elements in X. Consider the set $\mathbb{Z}[X^*] = \{n_1w_1 + \cdots + n_lw_l \mid w_i \in X^*, n_i \in \mathbb{Z}\}$. Let $v = n_1v_1 + \cdots + n_pv_p, u = m_1u_1 + \cdots + m_ku_k \in \mathbb{Z}[X^*]$. Consider the set $\{v_1, \dots, v_p, u_1, \dots, u_k\}$ and relabel elements of this set as $\{w_1, \dots, w_l\}$, renumber coefficients and pad out with 0's as necessary and get: $v = n_1w_1 + \cdots + n_lw_l$ and $u = m_1w_1 + \cdots + m_lw_l$. Define addition so that $\mathbb{Z}[X^*]$ by

$$(n_1w_1 + \cdots + n_lw_l) + (m_1w_1 + \cdots + m_lw_l) = (n_1 + m_1)w_1 + \cdots + (n_l + m_l)w_l.$$

We are defining $nx = \underbrace{x + x + \cdots + x}_{n-\text{times}}$. Define multiplication on $\mathbb{Z}[X^*]$ by

$$\left(\sum_{i=1}^{l} n_iw_i\right)\left(\sum_{j=1}^{k} m_ju_j\right) = \sum_{i=1}^{l}\sum_{j=1}^{k} n_im_jw_iu_j,$$

where w_iu_j is the concatenation of the words w_i and u_j. Notice that the empty word ε is the unit of $\mathbb{Z}[X^*]$. These operations meet the requirements for $\mathbb{Z}[X^*]$ to become a unital ring. We claim that $\mathbb{Z}[X^*]$ is the free unital ring on X.

Theorem 2.11. $\mathbb{Z}[X^*]$ *is the free unital ring on* X.

Proof. Let R be a ring with unit 1 and $\varphi : X \to R$ a map. We have the inclusion map $i : X \to \mathbb{Z}[X^*]$. Note that if w is a word on X, then we may write $w = x_1 \cdots x_n$, where $x_i \in X$. Let $\widehat{\varphi} : \mathbb{Z}[X^*] \to R$ be given by

$$\widehat{\varphi}\bigg(\sum_{i=1}^{l} n_i w_i \bigg) = \sum_{i=1}^{l} n_i \varphi(x_{i1}) \cdots \varphi(x_{im_i}),$$

where $w_i = x_{i1} \cdots x_{im_i}$ with $x_{ik} \in X$. In particular, $\widehat{\varphi}(\varepsilon) = 1$. It should be apparent that $\widehat{\varphi}$ is a homomorphism such that $\widehat{\varphi} \circ i = \varphi$. As for uniqueness, assume there is another such homomorphism $\widehat{\varphi}'$ such that $\widehat{\varphi}' \circ i = \varphi$. Then we have that $\widehat{\varphi} \circ i(x) = \varphi(x) = \widehat{\varphi}' \circ i(x)$ for all $x \in X$. Thus $\widehat{\varphi}$ and $\widehat{\varphi}'$ agree on a set of generators for $\mathbb{Z}[X^*]$ (since X generates this ring), so we deduce that $\widehat{\varphi} = \widehat{\varphi}'$. Hence, $\widehat{\varphi}$ is unique, and $\mathbb{Z}(X^*)$ is the free ring on X. □

Example 2.10. When $X = \emptyset$, $\mathbb{Z}[X^*] = \{n\varepsilon \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$. Let $X = \{x\}$. Then $\mathbb{Z}[X^*] = \{n_0\varepsilon + n_1 x + n_2 xx + \cdots + n_l xx \cdots x \mid l \geq 0; n_0, \ldots, n_l \in \mathbb{Z}\}$, which is isomorphic to $\mathbb{Z}[x]$, where $\mathbb{Z}[x]$ is the polynomial ring with indeterminate x and integer coefficients. In general, if $X = \{x_1, ..., x_n\}$, then $\mathbb{Z}[X^*]$ is isomorphic to the polynomial ring with noncommuting indeterminantes $x_1, ..., x_n$ and integer coefficients (often denoted $\mathbb{Z}\langle x_1, \ldots, x_n \rangle$).

2.5.2 Free Fields

Unfortunately, not every category has free objects. In this section, we show that free fields do not exist. Let X be a set. Assume that the free field on X exists. Call it $\mathbb{F}(X)$, and let i denote the corresponding map from X into $\mathbb{F}(X)$. Then for any map φ from X into a field $\mathbb{K}$, there exists a unique field homomorphism $\widehat{\varphi} : \mathbb{F}(X) \to \mathbb{K}$ such that $\widehat{\varphi} \circ i = \varphi$. There are two cases to consider.

If X is nonempty, let $\mathbb{F}(X)$ have characteristic m ($m = 0$ or m is prime). Let $\mathbb{K}$ be any field of any characteristic other than m. Then the only homomorphism from $\mathbb{F}(X)$ to $\mathbb{K}$ is the zero map. Forcing $\widehat{\varphi}$ to be the zero map implies that $\widehat{\varphi} \circ i = \varphi$ sends everything to zero as well. But we are free to let φ send elements of X to anything in $\mathbb{K}$ that we want. In particular, φ does not have to send everything to zero (contradiction).

If X is empty, then i and φ are the canonical empty functions. So the universal mapping property just asserts that for each field $\mathbb{K}$ there exists a unique homomorphism $\widehat{\varphi} : F(\emptyset) \to \mathbb{K}$ (no compatibility criterion needs to be met). But there are two homomorphisms from $\mathbb{F}(\emptyset)$ to itself, namely the zero morphism and the identity map. This contradicts the uniqueness requirement. Therefore, free fields cannot exist.

If we require that homomorphisms send the unit of the domain to the unit of the codomain, then we can modify this argument to notice that $\mathbb{F}(X)$ has no morphisms into a field of characteristic not m. Thus no free fields can exist.

3 Lie Algebras

In this section, we define the Lie algebra and examine its basic properties. We then move on to study the universal enveloping algebra in section 3.2, and the free Lie algebra in section 3.3.

3.1 Basic Properties

Definition 3.1. *A* ***Lie algebra*** L *over a field* $\mathbb{F}$ *is an algebra such that its multiplication is alternating and obeys the Jacobi identity. That is, we require that*

1. $[x, x] = 0$ *(alternating), and*
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *for all* $x, y, z \in L$ *(Jacobi identity).*

Recall from section 2.4.1 that an algebra's multiplication is by definition bilinear. The bilinear and alternating multiplication imply that for all x and y in L, $[x, y] = [-y, x]$. Indeed, we have

$$\begin{aligned} 0 &= [x + y, x + y]. \\ &= [x, x] + [x, y] + [y, x] + [y, y]. \\ &= [x, y] + [y, x]. \end{aligned}$$

Hence, $[x, y] = -[y, x]$. We call this property skew-symmetry. If the characteristic of our field is not two, then we can show that skew-symmetry implies alternating. If skew-symmetry holds, then

$$[x, x] = -[x, x]. \tag{1}$$

$$[x, x] + [x, x] = 0. \tag{2}$$

$$(1 + 1)[x, x] = 0. \tag{3}$$

If the characteristic of the field is not two, then we know that $1+1 \neq 0$, and so equation 3 implies that $[x, x] = 0$. If the characteristic of the field is 2, then $1+1=0$, and so we can not conclude that $[x, x] = 0$.

We say that a Lie algebra L is **abelian** if $[x, y] = 0$. Note that because of skew-symmetry, we have that $[x, y] = 0$ if and only if $[x, y] = [y, x]$. Note that all one dimensional Lie algebras are abelian.

Using skew-symmetry and bilinearity, we can rewrite the Jacobi identity.

$$
\begin{aligned}
[x,[y,z]] + [y,[z,x]] + [z,[x,y]] &= 0. \\
[x,[y,z]] &= -[z,[x,y]] - [y,[z,x]]. \\
&= [[x,y],z] + [y, -[z,x]]. \\
&= [[x,y],z] + [y,[x,z]].
\end{aligned}
$$

The equation $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$ is a more useful form of the Jacobi identity, for it reveals a structure similar to that of the product rule for derivatives. To see this, first define the left multiplication operator, $L_x(y) : L \to L$, to be given by $L_x(y) = [x, y]$. The operator simply multiplies a vector on the left by x. We rewrite the more useful form of the Jacobi in terms of the left multiplication operator.

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]].$$

$$L_x([y,z]) = [L_x(y), z] + [y, L_x(z)].$$

The left multiplication operator on the product $[y, z]$ behaves like differentiation on the product of two functions. We formalize this observation with the following definition.

Definition 3.2. *Let A be an algebra over a field $\mathbb{F}$. If $\partial : A \to A$ is a linear map such that*

$$\partial(x \cdot y) = \partial(x) \cdot y + x \cdot \partial(y),$$

then ∂ is a ***derivation*** *of A.*

The Jacobi identity implies that the left multiplication operator L_x is a derivation of a Lie algebra.

Example 3.1. Let $L = \mathbb{R}^3$. Let $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ denote the standard unit vectors. Define the multiplication by the vector cross-product rule: $[\mathbf{i}, \mathbf{j}] = \mathbf{k}$, $[\mathbf{j}, \mathbf{k}] = \mathbf{i}$, and $[\mathbf{k}, \mathbf{i}] = \mathbf{j}$. This makes L into a Lie algebra.

Example 3.2. Let $\mathcal{A}$ be any associative algebra. Then $\mathcal{A}$ is a Lie algebra under the **commutator bracket**, $[,] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, which is given by $[x, y] = x \cdot y - y \cdot x$, where $\cdot$ denotes the associative multiplication. The commutator bracket is a very useful way of turning any associative algebra into a Lie algebra. For example, the set of all $n \times n$ matrices is a Lie algebra under the commutator bracket since matrix multiplication is associative. We adopt the notation that if $\mathcal{A}$ is an associative algebra, then $[\mathcal{A}]$ denotes the Lie algebra obtained from $\mathcal{A}$ by equipping it with the commutator bracket.

Next, we define subalgebras and ideals.

Definition 3.3. *A subset S of a Lie algebra L is a* ***subalgebra*** *of L if*

1. *S is a subspace of L, and*

2. *$[S, S] \subseteq S$. That is, for all $x, y \in S$, $[x, y] \in S$.*

Definition 3.4. *A subset I of a Lie algebra L is an* ***ideal*** *of L if*

1. *I is a subspace of L, and*

2. *$[I, L] \subseteq I$. That is, for all $x \in I$ and $y \in L$, $[x, y] \in I$.*

Because of skew-symmetry, we have $[I, L] \subseteq I$ if and only if $[L, I] \subseteq I$. This makes it easier to check that a subspace of a Lie algebra is an ideal. This fact also applies to subalgebras. It follows from the definitions that all ideals are subalgebras, but not all subalgebras are ideals. Also, as with subspaces, the intersection of two ideals is an ideal, yet the union of two ideals is not an ideal unless one of the ideals contains the other.

Example 3.3. Let $gl(2, \mathbb{R})$ denote the vector space of all 2×2 matrices over $\mathbb{R}$. From example 2, because matrix multiplication is associative, we can turn $gl(2, \mathbb{R})$ into a Lie algebra by equipping it with the commutator bracket: $[x, y] = xy - yx$, where xy is the usual product of matrices x and y. If $b(2, \mathbb{R})$ is the set of all 2×2 upper triangular matrices, then $b(2, \mathbb{R})$ is a subalgebra of $g(2, \mathbb{R})$. To check that $b(2, \mathbb{R})$ is a subalgebra, one would need to verify that it is indeed a subspace of $g(2, \mathbb{R})$ and that the multiplication is closed. It suffices to check closure on any basis. So, if $\beta = \{e_{11}, e_{12}, I\}$, where I is the 2×2 identity matrix and e_{ij} is the matrix with a one in the i^{th} row and j^{th} column and zeros elsewhere, then observe that

$$[e_{11}, e_{12}] = e_{11}e_{12} - e_{12}e_{11} = e_{12} \in b(2, \mathbb{R});$$

$$[e_{11}, I] = e_{11}I - Ie_{11} = 0;$$
$$[e_{12}, I] = e_{12}I - Ie_{12} = 0.$$

We know all other multiplications are closed because of alternating and skew-symmetry. Note that $b(2, \mathbb{R})$ is not an ideal of $gl(2, \mathbb{R})$. To check this, we observe that although

$$y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in gl(2, \mathbb{R}),$$

the product

$$[e_{11}, y] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \notin b(2, \mathbb{R}).$$

Example 3.4. Let L be a Lie algebra over a field $\mathbb{F}$. We define the **center** of L to be the set $Z(L) = \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}$. We show that the center is an ideal of L. Observe that if $x_1, x_2 \in Z(L)$, $y \in L$, and $c \in \mathbb{F}$, then $[x_1 + cx_2, y] = [x_1, y] + c[x_2, y] = 0$. Hence, $Z(L)$ is a subspace of L. Next, we must show that if $x \in Z(L)$ and $z \in L$, then $[x, z] \in Z(L)$. The Jacobi identity implies that $[[x, z], y] = [x, [z, y]] - [z, [x, y]] = 0$. Hence, $Z(L)$ is an ideal of L.

Next, we define the homomorphism, kernel, and quotient space for Lie algebras.

Definition 3.5. *Let L_1 and L_2 be Lie algebras over a field $\mathbb{F}$. A linear map $\varphi : L_1 \to L_2$ is a **homomorphism** if it preserves the multiplication. That is,*

$$\varphi[x, y] = [\varphi(x), \varphi(y)] \text{ for all } x, y \in L_1.$$

Definition 3.6. *Let L_1 and L_2 be Lie algebras over a field $\mathbb{F}$. If φ is a homomorphism from L_1 into L_2, then the **kernel** of φ, denoted $ker(\varphi)$, is the set of all elements in L_1 whose image under φ is zero. That is,*

$$ker(\varphi) = \{x \in L_1 \mid \varphi(x) = 0\}.$$

Note that the kernel of a Lie algebra homomorphism is an ideal.

Definition 3.7. *Let L be a Lie algebra over a field $\mathbb{F}$ and I an ideal of L. The **quotient space** is defined as*

$$L/I = \{x + I \mid x \in L\}.$$

Addition and scalar multiplication are defined by

$$(x + I) + (y + I) = (x + y) + I,$$

and

$$c(x + I) = (cx) + I$$

for all $x, y \in L$ *and* $c \in \mathbb{F}$. *Multiplication on* L/I *is given by*

$$[x + I, y + I] = [x, y] + I.$$

Note that the multiplication on L/I bilinear since the multiplication on L is bilinear. To check that the multiplication is well-defined, let $x + I = x' + I$ and $y + I = y' + I$ for $x, x', y, y' \in L$. It follows that $x = x' + u$ and $y = y' + v$ for some $u, v \in I$. Therefore,

$$\begin{aligned}[x, y] - [x', y'] &= [x' + u, y' + v] - [x', y']. \\ &= [x', y'] + [x', v] + [u, y'] + [u, v] - [x', y']. \\ &= [x', v] + [u, y'] + [u, v] \in I\end{aligned}$$

because I is an ideal. Thus, $[x, y] + I = [x', y'] + I$, which implies that

$$[x + I, y + I] = [x, y] + I = [x', y'] + I = [x' + I, y' + I].$$

This shows that the multiplication is well-defined. We also have the canonical homomorphism $\varphi : L \to L/I$ given by

$$\varphi(x) = x + I$$

for all $x \in L$. We can readily verify that φ is a homomorphism since

$$\varphi(cx + y) = (cx + y) + I = c(x + I) + (y + I) = c\varphi(x) + \varphi(y),$$

and

$$\varphi([x, y]) = [x, y] + I = [x + I, y + I] = [\varphi(x), \varphi(y)]$$

for all $x, y \in L$ and $c \in \mathbb{F}$. In particular, this canonical homomorphism is actually an epimorphism.

The usual isomorphism theorems hold for Lie algebras. We make use of the first isomorphism theorem, so we restate it here.

Theorem 3.1. *Let* L_1 *and* L_2 *be Lie algebras over a field* $\mathbb{F}$ *and* $\varphi : L_1 \to L_2$ *be a homomorphism. Then* $L_1/ker(\varphi) \cong \varphi(L_1)$. *If* φ *is surjective, then* $L_1/ker(\varphi) \cong L_2$.

Now that we have established some of the basic properties of Lie algebras, we move our attention to universal enveloping algebras of Lie algebras.

3.2 The Universal Enveloping Algebra

Let L be a Lie algebra over a field $\mathbb{F}$. Let

$$T = \bigoplus_{n=0}^{\infty} T(L)^n = T(L)^0 \oplus T(L)^1 \oplus T(L)^2 \oplus \cdots$$

denote the tensor algebra of L. Recall from section 2.4.2 that $T(L)^0 = \mathbb{F}$, $T(L)^1 = L$, $T(L)^2 = L \otimes L$ and so on. Let J be the ideal of T generated by all elements of the form

$$x \otimes y - y \otimes x - [x, y],$$

where $x, y \in L$. The quotient space $\mathfrak{U}(L) = T/J$ is an associative algebra over $\mathbb{F}$ called the **universal enveloping algebra** of L.

Example 3.5. Let L be an n dimensional abelian Lie algebra over $\mathbb{F}$. Consequently, if $\beta = \{x_1, ..., x_n\}$ is a basis for L, then $[x_i, x_j] = 0$ for all $x_i, x_j \in \beta$. The ideal J is therefore generated by all elements of the form $x \otimes y - y \otimes x$ for $x, y \in L$. This implies that $\mathfrak{U}(L) = T/J$ is a commutative algebra. In particular, $\mathfrak{U}(L)$ is generated by β, and the identity $1 \in \mathbb{F}$, which implies $\mathfrak{U}(L)$ is also isomorphic to the polynomial algebra $\mathbb{F}[x_1, ..., x_n]$.

In general, we have the inclusion map $i : L \to T$, and the canonical projection map $\pi : T \to \mathfrak{U}(L)$. Let $\sigma : L \to \mathfrak{U}(L)$ be the composition of π with i. Note that σ is linear. Recall from example 3.2 that for an associative algebra A, $[A]$ denotes the Lie algebra obtained from A by equipping it with the commutator bracket. We proceed by showing that $\mathfrak{U}(L)$ satisfies a universal mapping property.

Theorem 3.2. *Let L be a Lie algebra over $\mathbb{F}$ and A be an associative algebra with unit 1 over $\mathbb{F}$, and $[A]$ the corresponding Lie algebra. Then given any Lie algebra homomorphism $\theta : L \to [A]$, there exists a unique associative algebra homomorphism $\phi : \mathfrak{U}(L) \to A$ such that $\phi \circ \sigma = \theta$.*

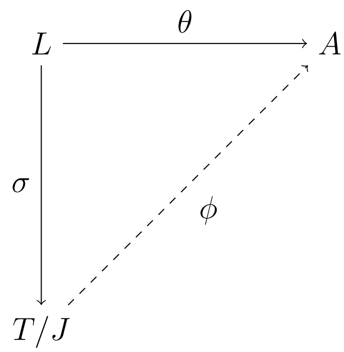

Figure 3: The commutative diagram for $\mathfrak{U}(L) = T/J$.

Proof. First, note that if $\beta = \{b_1, b_2, ...\}$ is a basis for L, then recall that $\mathbf{B} = \{b_{i_1} \otimes \cdots \otimes b_{i_n} \mid b_{i_k} \in \beta\}$ is a basis for T. Consider the map $\overline{\theta} : \mathbf{B} \to A$ given by

$$\overline{\theta}(b_{i_1} \otimes \cdots \otimes b_{i_n}) = \theta(b_{i_1}) \cdots \theta(b_{i_n}).$$

Because T is free on $\mathbf{B}$, this map can be extended to a unique associative algebra homomorphism θ' from T to A such that θ' and $\overline{\theta}$ agree on $\mathbf{B}$. Note that since $\beta \subset \mathbf{B}$, $\overline{\theta}$ and θ' must also agree on β, which means they agree on L. Let $x, y \in L$. Then

$$\begin{aligned}\theta'(x \otimes y - x \otimes y - [x, y]) &= \theta'(x \otimes y) - \theta'(y \otimes x) - \theta'([x, y]). \\ &= \overline{\theta}(x \otimes y) - \overline{\theta}(y \otimes x) - \overline{\theta}([x, y]). \\ &= \theta(x)\theta(y) - \theta(y)\theta(x) - \theta([x, y]). \\ &= 0\end{aligned}$$

where the last equality holds because $\theta : L \to [A]$ is a Lie algebra homomorphism. Thus, all the generators of the ideal J are in the kernel of θ'. Since the kernel is also an ideal, J must lie in the kernel of θ'. Therefore, there is an induced homomorphism $\phi : T/J \to A$ such that $\phi \circ \pi = \theta'$. If we restrict the domain to L, then we get that $\phi \circ \sigma = \theta$.

As for the uniqueness of ϕ, suppose that $\phi' : \mathfrak{U}(L) \to A$ is another such associative algebra homomorphism. Then $\phi \circ \sigma = \phi' \circ \sigma$. This implies that ϕ and ϕ' agree on $\sigma(L)$. Since L generates T, it follows that $\sigma(L)$ generates $\mathfrak{U}(L) = T/J$. Because ϕ and ϕ' agree on the generators of $\mathfrak{U}(L)$, they must also agree on $\mathfrak{U}(L)$, and so they are the same.

□

Like many of the objects in previous sections, the universal mapping property of universal enveloping algebras ensures that they are unique up to isomorphism. The next theorem provides us with a basis for $\mathfrak{U}(L)$.

Theorem 3.3 (*Poincaré-Birkhoff-Witt*)**.** *Let L be a Lie algebra with basis $\{x_i\ ;\ i \in I\}$. Let $<$ be a total order on the index set I. Let $\sigma : L \to \mathfrak{U}(L)$ be the natural linear map from L into its enveloping algebra. Let $\sigma(x_i) = y_i$. Then the elements*

$$y_{i_1}^{r_1} \otimes \cdots \otimes y_{i_n}^{r_n}$$

for all $n \geq 0$, all $r_i \geq 0$, and all $i_1, ..., i_n \in I$ with $i_1 < \cdots < i_n$ form a basis for $\mathfrak{U}(L)$.

For a proof of the Poincaré-Birkhoff-Witt theorem, see [CA]. We give an example of how to use the Poincaré-Birkhoff-Witt theorem to find a basis for the universal enveloping algebra of $\mathrm{sl}_2(\mathbb{F})$.

Example 3.6. Consider the Lie algebra $\mathrm{sl}_2(\mathbb{F})$ of all 2×2 matrices over $\mathbb{F}$ with trace 0. It has a basis given by $\{e, f, h\}$, where $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. It follows that a basis for the universal enveloping algebra of $\mathrm{sl}_2(\mathbb{F})$ is given by

$$\{1, e, f, h, e \otimes e, e \otimes f, e \otimes h, f \otimes f, f \otimes h, h \otimes h, ...\}.$$

Note the ordering of the basis elements. The Poincaré-Birkhoff-Witt theorem requires that we establish an ordering of the Lie algebra basis vectors, so in this example, we choose to order the basis for $\mathrm{sl}_2(\mathbb{F})$ so that $e < f < h$. Note that although $\mathrm{sl}_2(\mathbb{F})$ is finite-dimensional, its universal enveloping algebra is infinite dimensional. This is true in general for all nontrivial Lie algebras. That is, the universal enveloping algebra of a Lie algebra is always infinite-dimensional, so long as the Lie algebra is not the zero algebra. Also, it is convenient to use juxtaposition as opposed to tensors when multiplying elements in the universal enveloping algebra (we are justified in doing this since tensors are associative.) We can therefore rewrite the basis as

$$\{1, e, f, h, e^2, ef, eh, f^2, fh, h^2, ...\}.$$

We prove two corollaries of the Poincaré-Birkhoff-Witt theorem.

Corollary 3.3.1. *The map $\sigma : L \to \mathfrak{U}(L)$ is injective.*

Proof. Let $\{x_i \ ; \ i \in I\}$ be a basis for L. Suppose that $v \in \ker(\sigma)$. Then $\sigma(v) = 0$ and we have

$$\begin{aligned}\sigma(v) &= \sigma\left(\sum_{i=1}^{n} c_i x_i\right) \text{ for some scalars } c_1, ..., c_n. \\ &= \sum_{i=1}^{n} c_i \sigma(x_i). \\ &= \sum_{i=1}^{n} c_i y_i = 0.\end{aligned}$$

Because each of the y_i are linearly independent, it must be the case that $c_1 = \cdots = c_n = 0$, which implies that $v = 0$. So, the kernel of σ is zero. □

Corollary 3.3.2. *The subspace $\sigma(L)$ is a Lie subalgebra of $[\mathfrak{U}(L)]$ isomorphic to L. Thus, σ identifies L with a Lie subalgebra of $[\mathfrak{U}(L)]$.*

Proof. By corollary 3.3.1, we know that $\sigma : L \to \sigma(L)$ is bijective, and so $L \cong \sigma(L)$. We know that $\sigma(L)$ is a subspace of $\mathfrak{U}(L)$ since it is the range of a linear transformation. To check closure under the bracket, note that the elements $y_i, i \in I$, form a basis of $\sigma(L)$ and that

$$[y_i, y_j] = [\sigma(x_i), \sigma(x_j)] = \sigma([x_i, x_j]).$$

Hence, $[y_i, y_j] \in \sigma(L)$ and so $\sigma(L)$ is a Lie subalgebra of $[\mathfrak{U}(L)]$. □

The universal enveloping algebra has many uses in the theory of Lie algebras. For our purposes, the universal enveloping algebra of a Lie algebra helps study free Lie algebras.

3.3 Free Lie Algebras

Let X be a set, and $V(X)$ the vector space with basis X. Consider the tensor algebra $F(X)$ of $V(X)$. Recall that $F(X)$ is the free associative algebra on X. Since $F(X)$ is associative, let $[F(X)]$ be the corresponding Lie algebra obtained from $F(X)$ by equipping it with the commutator bracket. Note that X is a subset of $[F(X)]$. Let $FL(X)$ be the intersection of all Lie subalgebras of $[F(X)]$ that contain X. That is, $FL(X)$ is the subalgebra of $[F(X)]$ generated by the X. We show that $FL(X)$ is the free Lie algebra on X.

Theorem 3.4. *$FL(X)$ is the free Lie algebra on the set X.*

Proof. We have the inclusion map $i : X \to FL(X)$. Let L be a Lie algebra and $\theta : X \to L$ a map. We have the composition $\theta' = \sigma \circ \theta : X \to \mathfrak{U}(L)$, where σ is the natural linear map from the Lie algebra L into its universal enveloping algebra $\mathfrak{U}(L)$. Since X is basis for the vector space $V(X)$, θ' can be extended to a unique linear map $\phi : V(X) \to \mathfrak{U}(L)$ such that ϕ and θ' agree on X. Because $V(X)$ is a basis for the tensor algebra $F(X)$ of $V(X)$, ϕ can be extended to a unique associative algebra homomorphism $\phi' : F(X) \to \mathfrak{U}(L)$ such that ϕ' and ϕ agree on $V(X)$. This same map gives a Lie algebra homomorphism from $[F(X)]$ to $[\mathfrak{U}(L)]$. We also see that ϕ' and θ' agree on X and so $\phi'(X) = \theta'(X) = \sigma \circ \theta(X)$. This implies that $\phi'(X) \subset \sigma(L)$. We know from corollary 3.3.2 that $\sigma(L)$ is a Lie subalgebra of $[\mathfrak{U}(L)]$ isomorphic to L. The set $S = \{x \in [F(X)] \mid \phi'(x) \in \sigma(L)\}$ is also a subalgebra of $[F(X)]$. In particular, S contains X, which means it also contains $FL(X)$. Therefore, by restricting the domain, we obtain $\phi' : FL(X) \to \sigma(L)$. Since the map $\sigma : L \to \sigma(L)$ is bijective, we are justified in defining $\varphi : FL(X) \to L$ by $\varphi = \sigma^{-1} \circ \phi'$. We check that φ and θ agree on X by observing that for $x \in X$, we have

$$\begin{aligned}
\varphi \circ i(x) &= \sigma^{-1} \circ \phi' \circ i(x). \\
&= \sigma^{-1} \circ \phi(x). \\
&= \sigma^{-1} \circ \theta'(x). \\
&= \sigma^{-1} \circ \sigma \circ \theta(x). \\
&= \theta(x).
\end{aligned}$$

Thus, we have a homomorphism of the required type. To show that φ is unique, let φ' be another such homomorphism. Let S' be the set of elements in $FL(X)$ for which φ and φ' agree. Because $\varphi \circ i = \varphi' \circ i$, we see that φ and φ' agree on X. Moreover, S' is the kernel of the homomorphism $(\varphi - \varphi')$, so it is necessarily a subalgebra of $FL(X)$. Since X generates $FL(X)$ as a Lie subalgebra, we conclude that φ and φ' agree on $FL(X)$. □

Example 3.7. Not surprisingly, if $X = \emptyset$, then $FL(\emptyset) = \{0\}$. If X is a singleton set, say $X = \{x\}$, then the free Lie algebra on X is given by $FL(X) = \mathrm{span}\{x\}$, which is a one dimensional abelian Lie algebra (there is only one such algebra up to isomorphism). If $|X| \geq 2$, then $FL(X)$ is infinite dimensional.

The next theorem illustrates that we may identify the free associative algebra on a set X with the universal enveloping algebra of the free Lie algebra on X.

Theorem 3.5. *The universal enveloping algebra* $\mathfrak{U}(FL(X))$ *is isomorphic to* $F(X)$.

Proof. We show that $F(X)$ is also a universal enveloping algebra for the free Lie algebra $FL(X)$. We do this by showing that $F(X)$ satisfies the universal mapping property given in theorem 3.2. So, let A be an associative algebra with unit 1 and $\theta : FL(X) \to [A]$ be a Lie algebra homomorphism. We have the inclusion map $\sigma : FL(X) \to F(X)$. Restricting the domain of σ, we obtain a map from X to $F(X)$. Observe that since $F(X)$ is the free associative algebra on X, there exists a unique associative algebra homomorphism $\phi : F(X) \to A$ such that ϕ and θ agree on X. This same map gives a Lie algebra homomorphism from $[F(X)]$ to $[A]$. Since X generates $FL(X)$ as a Lie subalgebra, and $FL(X) \subset F(X)$, we deduce that ϕ and θ must also agree on $FL(X)$. As for uniqueness, recall that ϕ is the unique homomorphism from $F(X)$ to A such that ϕ and θ agree on X. Since $X \subset FL(X)$, it must also be that ϕ is the unique homomorphism from $F(X)$ to A such that ϕ and θ agree on $FL(X)$. □

One may use what are called Lyndon words to construct a basis for the free Lie algebra, called the Lyndon basis. Lyndon bases are not easily accessible without a working knowledge of combinatorics. Therefore, we do not discuss them in this paper. We proceed by discussing Leibniz algebras.

4 Leibniz Algebras

Leibniz algebras are a generalization of Lie algebras. In this section, we begin with the basis properties of Leibniz algebras and conclude with free Leibniz algebras.

4.1 Basic Properties

Definition 4.1. *A* ***left Leibniz algebra*** A *is an algebra that obeys the Leibniz identity. That is, we require that*

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]] \text{ for all } x,y,z \in A. \quad (\textit{Leibniz Identity})$$

Note that the Leibniz identity is actually just the more useful form of the Jacobi identity that we derived in section 3.1. It follows that all Lie algebras are Leibniz algebras. What separates the two is that a Leibniz algebra's multiplication is not necessarily alternating. Consequently, Leibniz algebras do not necessarily have skew-symmetry either. This requires that we distinguish between left and right Leibniz algebras. We make this distinction by noticing that the Leibniz identity in definition 4.1 shows that the left multiplication operator L_x, which is given by $L_x(y) = [x, y]$, is actually a derivation of A. Therefore, a left Leibniz algebra is an algebra whose left multiplication operator is a derivation. Similarily, a right Leibniz algebra is an algebra whose right multiplication operator R_x, which is given by $R_x(y) = [y, x]$, is a derivation of A. Like Lie algebras, we say that a Leibniz algebra is **abelian** if $[x, y] = 0$ for all $x, y \in A$.

Example 4.1. Let A be a two dimensional algebra over a field $\mathbb{F}$ with multiplications given by

$$[x, y] = [x, x] = y \text{ and } [y, x] = [y, y] = 0.$$

Then A is a non-Lie, left Leibniz algebra because the left multiplication operator is a derivation. However, note that

$$R_y([x, x]) = [[x, x], y] = 0,$$

yet

$$[R_y(x), x] + [x, R_y(x)] = [[x, y], x] + [x, [x, y]] = y \neq 0.$$

Thus, A is not a right Leibniz algebra because the right multiplication operator is not a derivation.

Example 4.2. Let A be a two dimensional algebra over a field $\mathbb{F}$ with multiplications given by

$$[x, x] = [y, x] = y \text{ and } [x, y] = [y, y] = 0.$$

Then A is a non-Lie, right Leibniz algebra. However, note that

$$L_x([x, x]) = [x, [x, x]] = 0,$$

yet

$$[L_x(x), x] + [x, L_x(x)] = [[x, x], x] + [x, [x, x]] = y \neq 0.$$

Thus, A is not a left Leibniz algebra.

Let A be a left (right) Leibniz algebra and $a \in A$. Then we define a^n inductively by letting $a^1 = a$ and $a^{n+1} = [a, a^n]$ ($a^{n+1} = [a^n, a]$). Using induction, one can show that if $k \geq 2$, then $L_{a^k} = 0$ ($R_{a^k} = 0$), where the base case is a direct consequence of the Leibniz identity. That is, for left Leibniz algebras, left multiplication by a an element of power greater than 1 yields zero, and for right Leibniz algebras, right multiplication by an element of power greater than 1 yields zero. Next, we give an example of a Leibniz algebra that both a left and a right Leibniz algebra.

Example 4.3. Let A be a two dimensional algebra over a field $\mathbb{F}$ generated by a single element. That is, $A = \text{span}\{x, x^2\}$. Let the multiplications be given by

$$[x^2, x] = [x, x^2] = 0.$$

Then A is both a left and a right Leibniz algebra. In particular, we say that A is a cyclic, nilpotent Leibniz algebra.

Since left Leibniz algebras are analogs of right Leibniz algebras, for the remainder of this paper, when we refer to a Leibniz algebra, we will assume it is a left Leibniz algebra. Subalgebras for Leibniz algebras are defined in the same way they are for Lie algebras. However, since we are not guaranteed skew-symmetry, we must distinguish between left and right ideals of Leibniz algebras.

Definition 4.2. *Let A be a Leibniz algebra over $\mathbb{F}$ and I a subspace of A. Then I is a* ***left ideal*** *if $[A, I] \subseteq I$, and I is a* ***right ideal*** *if $[I, A] \subseteq I$. If I is both a left and a right ideal, then we say I is an* ***ideal*** *of A.*

Example 4.4. Let A be the two-dimensional Leibniz algebra over $\mathbb{F}$ with $[y, x] = x$ and all other multiplications zero. Then $I = \text{span}\{x\}$ is an ideal of A, for it is both a left and a right ideal. Although $J = \text{span}\{y\}$ is a left ideal, it is not a right ideal since $[y, x] = x \notin J$.

Example 4.5. Let A be a Leibniz algebra over a field $\mathbb{F}$. We define the $\text{Leib}(A) = \text{span}\{[a, a] \mid a \in A\}$. The $\text{Leib}(A)$ is an ideal of A. To see why, let $a, b \in A$ and note that $[[b, b], a] = 0$, which implies that $\text{Leib}(A)$ is a right ideal. Keeping in mind that left multiplication by a power greater than 1 is zero, observe that

$$\begin{aligned} [a, [b, b]] &= [a, a] + [a, [b, b]] + [[b, b], a] + [[b, b], [b, b]] - [a, a]. \\ &= [a + [b, b], a + [b, b]] - [a, a]. \\ &\in \text{Leib}(A). \end{aligned}$$

Hence, Leib(A) is both a left and a right ideal, so we conclude that Leib(A) is an ideal of A. The significance of the Leib(A) is that it is the smallest ideal such that $A/\text{Leib}(A)$ is a Lie algebra. To see why, suppose that I is another ideal of A such that A/I is Lie. Then for all $y \in A$, $I = [y+I, y+I] = [y, y]+I$, which implies that $[y, y] \in I$. Hence, $\text{Leib}(A) \subseteq I$.

Now that we have established some of the basic properties of Leibniz algebras, we move our focus to free Leibniz algebras.

4.2 Free Leibniz Algebras

As usual, we begin with a set X. Recall from section 2.4.3 that we have the free non-associative algebra $\mathcal{A}$ on X. Let J be the ideal of $\mathcal{A}$ generated by elements of the form

$$[\mathbf{x}, [\mathbf{y}, \mathbf{z}]] - [[\mathbf{x}, \mathbf{y}], \mathbf{z}] - [\mathbf{y}, [\mathbf{x}, \mathbf{z}]],$$

where $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ are non-associative words on X. Then the quotient space $A(X) = \mathcal{A}/J$ is a Leibniz algebra, called the free Leibniz algebra on X. We show that $A(X)$ satisfies the universal mapping property for free objects.

Theorem 4.1. *The Leibniz algebra $A(X)$ is the free Leibniz algebra on X.*

Proof. Let A_1 be a Leibniz algebra and $\varphi : X \to A_1$ a map. We have the inclusion map $i : X \to A$, and the projection map $\pi : \mathcal{A} \to \mathcal{A}/J$. Let $\sigma = \pi \circ i : X \to \mathcal{A}/J$. We need a unique homomorphism $\widehat{\varphi} : \mathcal{A}/J \to A_1$ such that $\widehat{\varphi} \circ \sigma = \varphi$. Because $\mathcal{A}$ is the free non-associative algebra on X, there is a unique algebra homomorphism $\varphi' : \mathcal{A} \to A_1$ such that φ' and φ agree on X. Observe that for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{A}$, we have that

$$\begin{aligned}
&\varphi'\big[[\mathbf{x}, [\mathbf{y}, \mathbf{z}] - [[\mathbf{x}, \mathbf{y}], \mathbf{z}] - [\mathbf{y}, [\mathbf{x}, \mathbf{z}]]\big] \\
&\quad = [\varphi'(\mathbf{x}), [\varphi'(\mathbf{y}), \varphi'(\mathbf{z})]] - [[\varphi'(\mathbf{x}), \varphi'(\mathbf{y})], \varphi'(\mathbf{z})] - [\varphi'(\mathbf{y}), [\varphi'(\mathbf{x}), \varphi'(\mathbf{z})]]. \\
&\quad = 0.
\end{aligned}$$

The last equality holds because A_1 is a Leibniz algebra. Thus, we see that the generators of J lie in the kernel of φ'. Since the kernel is an ideal, we we deduce that $J \subseteq \ker(\varphi')$. This implies there is an induced homomorphism $\widehat{\varphi} : \mathcal{A}/J \to A_1$. Since $\widehat{\varphi}$ and φ agree on X, we deduce that $\widehat{\varphi} \circ \sigma = \varphi$. Because X generates $\mathcal{A}$, we know that the homomorphism is unique. □

Our ultimate result shows how to find a basis for the free Leibniz algebra. Although certain combinatorial methods are required to give an extensive treatment of free Leibniz algebras, we can, unlike free Lie algebras, at least understand what the free Leibniz algebra basis looks like without a working knowledge of combinatorics.

Theorem 4.2. *Let $A(X)$ be the free Leibniz algebra on X. Let β be the set containing elements of the form*

$$[x_1, [x_2, ..., [x_{n-1}, x_n]...]],$$

where $x_1, ..., x_n \in X$. Then β forms a basis for $A(X)$.

We prove that the basis spans the free Leibniz algebra. Showing linear independence requires combinatorial techniques that are outside the scope of this paper. For a complete proof, see [MI].

Proof (spanning set.) Recall from section 2.4.3 that we may define the length of a non-associative word on X as the number of elements of X used to construct the word. For example, the length of x is 1, the length of $[[x, x], z]$ is 3, and so on. We only consider non-empty words since the theorem trivially holds for empty words. Any non-associative word of length 1 or 2 is in $\text{span}(\beta)$. As for words of length 3, the Leibniz identity, when re-written as $[[x, y], z] = [x, [y, z]] - [y, [x, z]]$, ensures that all words of length 3 are elements of the span of β. We proceed by induction on the lengths of the words. Assume that any word of length less than n is an element of the span of β. Let $[A, B]$ represents a word of length n. We must show that $[A, B] \in \text{span}(\beta)$.

Suppose the length of A is 1. Then let $A = x_1$. The length of B must be less than n, so the inductive hypothesis implies $B = [x_2, [x_3, ...[x_{n-1}, x_n]]...]$. Therefore, $[A, B] = [x_1, [x_2, ...[x_{n-1}, x_n]]...]$.

We make a second inductive argument by assuming all elements of the form $[C, D]$ are in the span of β, where the length of $[C, D] = n$, and the length of C is less than k (which is also necessarily less than n). Suppose A has length k. Then k must be less than n, so by our first inductive hypothesis $A \in \text{span}(\beta)$. Let

$$A = \sum_{i=1}^{m} [x_{i_1}, [x_{i_2}, [...[x_{i_{k-1}}, x_{i_k}]]...].$$

From bilinearity, we have

$$[A, B] = \sum_{i=1}^{m} [[x_{i_1}, [x_{i_2}, [...[x_{i_{k-1}}, x_{i_k}]]...], B].$$

For notational purposes, let $C = [x_{i_2}, [...[x_{i_{k-1}}, x_{i_k}]]...]$. Then we can write $[A, B] = [[x_{i_1}, C], B] = [x_{i_1}, [C, B]] - [C, [x_{i_1}, B]]$. Notice that the length of $[C, B] < n$. So, the first inductive hypothesis implies $[x_{i_1}, [C, B]] \in \text{span}(\beta)$. As for $[C, [x_{i_1}, B]]$, note the length of C equals the length of $A-1 = k-1 < k$. Thus, the second inductive hypothesis implies $[C, [x_{i_1}, B]] \in \text{span}(\beta)$. Hence, $[A, B] \in \text{span}(\beta)$. □

Example 4.6. Let $X = \{x\}$. The set $A(X) = \text{span}\{x, x^2, x^3, ...\}$ is the free Leibniz algebra on the set X. We see that the free Leibniz algebra generated by one element is an infinite dimensional cyclic algebra. In general, all free Leibniz algebras are infinite dimensional with the only exception being the case where the generating set X is empty (in such a case $A(\emptyset) = \{0\}$).

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